

ABSTRACTS OF
ORAL and POSTER PRESENTATIONS

made at the

Twelfth International
Permutation Patterns Conference

East Tennessee State University

July 7–11, 2014

AMENDED ABSTRACT

Patterns and Statistics in Restricted Growth Functions

by

Samantha Dahlberg (Michigan State University)

Robert Dorward (Oberlin College)

Jonathan Gerhard (James Madison University)

Thomas Grubb (Michigan State University)

Carlin Purcell (Vassar College)

Lindsey Reppuhn (Kalamazoo College)

Bruce E. Sagan (Michigan State University)

Let $w = a_1a_2 \dots a_n$ be a word (sequence) of positive integers. The standardization of w is the word obtained by replacing all occurrences of the smallest integer in w by 1, all copies of the next smallest by 2, and so on. We say that w contains a word v as a pattern if there is a subword w' of w which standardizes to v . Otherwise w avoids v .

Call $w = a_1a_2 \dots a_n$ a restricted growth function (RGF) if $a_1 = 1$ and for all $i \geq 2$ we have $a_i \leq 1 + \max\{a_1, \dots, a_{i-1}\}$. RGFs are of interest because they are in bijection with set partitions. Let $\mathcal{R}_n$ denote the set of all restricted growth functions of length n . And given an RGF v we let $\mathcal{R}_n(v) = \{w \in \mathcal{R}_n : w \text{ avoids } v\}$. We should note that this notion of pattern avoidance is related to, but not the same as, the one for set partitions presented in the previous talk.

Wachs and White introduced four fundamental statistics on set partitions via their RGFs. For example, one such statistic is $lb(w) = \sum_i lb_i(w)$ where $lb_i(w)$ is the number of integers to the left of w_i which are also bigger than w_i . Letting q be a variable, we study the generating functions defined by $LB_n(v) = \sum_{w \in \mathcal{R}_n(v)} q^{lb(w)}$ for various RGFs v , as well as the analogous polynomials for the other statistics. In particular, we determine their coefficients, degrees, and other properties.

WORD-REPRESENTABLE GRAPHS AND PERMUTATION PATTERNS

SERGEY KITAEV
UNIVERSITY OF STRATHCLYDE

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that distinct letters x and y *alternate* in w if and only if (x, y) is an edge in E . For example, the 4-cycle labeled by 1, 2, 3 and 4 in clockwise direction, can be represented by the word 13243142.

Word-representable graphs generalize several classical classes of graphs, e.g. 3-colorable, comparability and circle graphs. Some basic questions to ask about these graphs are:

- Which graphs are word-representable?
- How many graphs on n vertices are word-representable?
- What is the minimum length of a word that represents a given graph?

In this talk, I will not only give a comprehensive introduction to the exciting theory of word-representable graphs, but also discuss its relevance to patterns in permutations and words.

Inversion Sequences

Carla D. Savage
Department of Computer Science
North Carolina State University
Raleigh, NC 27695-8206

Abstract

An inversion sequence is a sequence of positive integers $(e_1, \dots, e_n)$ satisfying $e_k < k$. Inversion sequences are used in various ways to encode permutations - for example, Lehmer codes and inversion tables.

We consider a generalization. An s -inversion sequence $(e_1, \dots, e_n)$ satisfies $0 < e_k < s_k$, where $s = (s_1, \dots, s_n)$ is an arbitrary sequence of positive integers. These s -inversion sequences are related to *lecture hall partitions* through Ehrhart Theory. Lecture hall partitions are integer partitions constrained by bounds on the ratios of consecutive parts. They were introduced in 1997 by Bousquet-Mélou and Eriksson, who showed a surprising relationship between the number of lecture hall partitions of an integer and the number of partitions of that same integer into odd parts.

In this talk we consider statistics on s -inversion sequences motivated by the theory of lecture hall partitions. We will give examples of the surprising power of s -inversion sequences to model statistics on a variety of combinatorial families, with some applications and open questions. We also present the results of some recent experiments that suggest conjectures about pattern-avoidance in inversion sequences.

Deflatability of Permutation Classes

Michael Albert (Otago), with Mike Atkinson (Otago),
Cheyne Homberger (UF) and Jay Pantone (UF)

A permutation class, $\mathcal{C}$, is *deflatable* if there is a proper subclass of $\mathcal{C}$ which contains all the simple permutations of $\mathcal{C}$. Recently, deflatability has been a useful tool in enumerating or determining the structure of some permutation classes (in particular some of those defined by avoiding two patterns of length four). So it seems natural to turn to principal classes and ask: *for which permutations π is $\text{Av}(\pi)$ deflatable?* We provide a partial answer to this question.

A permutation (in one line notation) is $\oplus$ -decomposable if it can be written as a concatenation of two non empty sequences, the terms of the second all being larger than the terms of the first. In this case we write $\pi = \alpha \oplus \beta$ where α and β are also permutations whose patterns match those of the decomposing sequences. There is a dual notion of $\ominus$ -decomposability, and all the results we state apply to this case as well by symmetry. A permutation is plus-reducible if it contains a consecutive pair of the form $i, i + 1$ and minus-reducible if it contains a consecutive pair of the form $i + 1, i$.

Theorem 1. *If $\pi = \alpha \oplus \beta$ where both α and β have length at least two, then $\text{Av}(\pi)$ is not deflatable.*

The only remaining case for decomposable π up to symmetry is $\pi = 1 \oplus \beta$. Here we have:

Theorem 2. *If $\pi = 1 \oplus \rho$ and ρ has length at least three and is not both plus and minus reducible then $\text{Av}(\pi)$ is not deflatable.*

However, aside from the known results for π of length three we cannot make any significant inroads on the remaining cases. We can show:

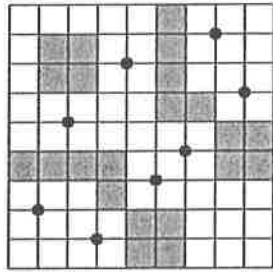
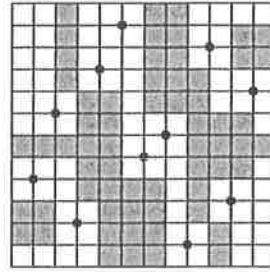
Theorem 3. *If π is of length four, then $\text{Av}(\pi)$ is not deflatable.*

In length five, $\text{Av}(\pi)$ is not deflatable for all decomposable π . Among the indecomposable cases all known examples are not deflatable, but the status of $\text{Av}(\pi)$ for $\pi \in \{25314, 23514, 24153, 24513\}$ (and symmetries) is unknown. From length six, we have examples of deflatable classes $\text{Av}(\pi)$ where π is either indecomposable (246135 for instance), or of the form $1 \oplus \beta$ (134652). Some other

cases not covered by our theorems remain open, though we do have examples of an infinite family of permutations π for which $\text{Av}(\pi)$ is deflatable.

We will discuss the methodology behind the proofs of the theorems on non-deflatability, which all rely on showing that, under the given conditions, one point extensions of non simple permutations in $\text{Av}(\pi)$ can always be found that are “more simple” than the original in the sense that the sum of “length minus one” over their maximal proper subintervals decreases strictly.

For the deflatable examples we construct reducible elements of $\text{Av}(\pi)$ in such a way that the two element interval they contain can never be split, except locally, in an extension in $\text{Av}(\pi)$. For instance, the following diagrams show respectively permutations in $\text{Av}(251364)$ and $\text{Av}(246135)$ which cannot be extended to simple permutations lying in those classes.



Collatz meets Fibonacci

Michael Albert

A well-known conjecture due to Lothar Collatz from 1937 states that when the function

$$f(x) = \begin{cases} x/2 & \text{if } x \equiv 0 \pmod{2} \\ 3x + 1 & \text{if } x \equiv 1 \pmod{2} \end{cases}$$

is iterated on an initial positive integer x we eventually reach the cycle $(1, 4, 2)$. Although many have tried to prove the conjecture, or find a counterexample to it, it remains open to this day. For this work we assume that the conjecture is true. The collection [1] contains many articles related to the conjecture and its generalizations, and an annotated bibliography. One of the intriguing features of this function is that, starting from some initial value x , the sequence of iterates: $x, f(x), f(f(x)), \dots$, behaves at first irregularly before its eventual apparent inevitable decline from some power of 2 down to the three element cycle. Here we consider the initial interesting phase of these iterations when seen from a distance, i.e. only with regard to their relative values, and not the actual numbers produced.

For example, starting the iteration at 12 provides the sequence:

$$\underbrace{12, 6, 3, 10, 5}_{\text{trace}}, 16, 8, 4, 2, 1, 4, 2, 1, \dots$$

The sequence of numbers up until the first power of two is the interesting phase of the iteration, which we will call its *trace*. The elements of a trace are all distinct, and viewing it from a distance we might replace each element of a trace by its rank (i.e. the i^{th} smallest number of the trace is replaced by i). The resulting sequence is a permutation, and we shall call permutations produced in this manner *Collatz permutations*. We denote the Collatz permutation obtained from initial value x by $C(x)$. So we have: $C(12) = 53142$. It seems natural to ask: *Among the permutations of length n , how many are Collatz permutations?*

Considering only those $x \leq 10^8$ for which the length of $C(x)$ is at most 7 produces Table 1. As the reader will have noticed the values in the table are the Fibonacci numbers. In what follows we will explain this phenomenon and show that it persists through length 14. Beyond that point, excess permutations appear – and we will explain how and why this occurs as well.

1 Types of traces

The appearance of the Fibonacci numbers in the enumeration of Collatz permutations is easy to explain. The steps that occur in a trace can be: *up steps*

Length	Collatz permutations
1	1
2	1
3	2
4	3
5	5
6	8
7	13

Table 1: Number of Collatz permutations of length less than 8 (experimentally).

($x \mapsto 3x + 1$ when x is odd) denoted by u ; and *down steps* ($x \mapsto x/2$ when x is even) denoted by d . Two up steps can never occur consecutively since $3x + 1$ is even when x is odd. The step types in a trace can be recovered from the resulting Collatz permutation according to the pattern of rises and descents. We call the resulting sequence of u 's and d 's the *type* of the trace. As well as not containing consecutive u 's, the last symbol in such a sequence must be a d (since there is a "hidden" u occurring next to take us to a power of 2). As is well known, the number of such sequences of length n is given by F_n , the n^{th} Fibonacci number (with $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$).

So, in order to show that there are at least F_n Collatz permutations of length n it will be enough to show that any sequence of u 's and d 's satisfying the necessary conditions above actually occurs as the type of some trace. To that end let a *witness* for a type, σ , be an $A = 2^a$ such that there is a trace ending at $(A - 1)/3$ with type σ .

The following proposition shows that every potential type has a witness and thereby proves that there are at least F_n Collatz permutations of length n .

Proposition 1. *If a type σ contains k u 's then there is a single congruence of the form $A \equiv c \pmod{3^{k+1}}$ which must be satisfied in order that a trace of type σ ends with witness A . Consequently, there is a least witness $A = 2^a$ with $a \leq 2 \cdot 3^k$, and a general witness is of the form 2^{a+jd} where j is a nonnegative integer and $d = 2 \cdot 3^k$.*

Table 2 shows that there are exactly F_n Collatz permutations of length n for $n = 1, 2, \dots, 14$ but for greater n there are more.

The first type that is associated with more than one Collatz permutation is the type $\sigma = uddudududdudd$ which has the integrality condition $2^a \equiv 16 \pmod{729}$. The smallest solution to this equation is $a = 4$ corresponding to the trace 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5 and the first permutation in Figure 1. However, the next solution to the integrality condition is $a = 490$, giving an initial number with 440 digits. This initial number produces a different permutation than the smaller initial number, and is shown in the second line in Figure 1.

In the next section we will explain why this type gives us two different permutations and when this should be expected.

length	#perms	length	#perms	excess
1	1	15	611	1
2	1	16	989	2
3	2	17	1600	3
4	3	18	2587	3
5	5	19	4185	4
6	8	20	6771	6
7	13	21	10953	7
8	21	22	17720	9
9	34	23	28669	12
10	55	24	46383	15
11	89	25	75044	19
12	144	26	121417	24
13	233	27	196448	30
14	377	29	317850	39

Table 2: Number of Collatz permutations of length less than 30

③	12	7	2	10	5	13	8	15	11	6	14	9	④	1
④	12	7	2	10	5	13	8	15	11	6	14	9	③	1

Figure 1: Two different permutations associated with the type *uddudududdudd*

2 Excess permutations

How can one type correspond to different permutations? We show that a type σ with n symbols gives rise to $n + 1$ linear functions. For example the type $\sigma = \text{dududd}$ gives the lines in Figure 2. We show that a witness for the type corresponds to wherever we find a vertical line $t = A$ where $A = 2^a$ and all the intersection points of $t = A$ with these lines are at integer heights. We can then read off the relative order of the lines at this point according to the order they are crossed as we move up the line $t = A$ from the t -axis.

Consequently we see that if we have two potential witnesses such that there is no intersection point between two lines in the corresponding family lying between them, that they will determine the same permutation. On the other hand, if there were a witness between every pair of intersections of the lines for a type, and if these intersections all occurred at distinct abscissae, we might have up to $\binom{n}{2}$ witnesses for any given type producing distinct permutations. However, we can rule out such a wealth of witnesses quite easily as we can show that the second potential witness always lies to the right of the rightmost intersection point among the lines. That is:

Proposition 2. *For any type σ there are at most two distinct permutations $C(x)$ arising from x of type σ .*

Propositions 1 and 2 show that the number of Collatz permutations of length n lies between F_n and $2F_n$.

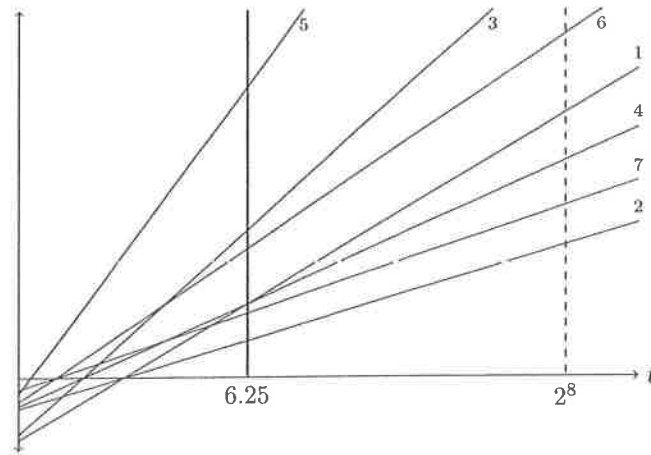


Figure 2: Linear functions determining permutations associated with the type *dududd*. The labels on each line correspond to the point that the corresponding element would occur in a trace and so the permutation associated with the intersections on the line $x = 2^8$ is 4 1 6 3 7 5 2.

To get an exact enumeration of the Collatz permutations one would need to understand which types are associated with two permutations. We call these *excess creating types* (ET's). Given an ET we can always create a new ET by prepending a *d*. This is because the extra *d* does not alter the integrality condition and can only increase the maximum intersection point. This at least shows that the number of ET's is non-decreasing.

As with so many aspects of the whole Collatz disease, a few answers just seem to lead to more questions.

- How exactly does the number e_n of ET's of length n behave? The data above suggests that it might be something like "half Fibonacci rate" i.e. $e_n \sim e_{n-2} + e_{n-4}$.
- In our current dataset we always get $c = 16$ in the integrality conditions for ET's. Is this just bias in the data that we currently have? Is it a necessary condition? Or sufficient?
- We have an extrinsic way of creating the Collatz permutations: run the Collatz process and see what comes out. Is there an intrinsic way to recognize these permutations, beyond the obvious condition that they cannot contain consecutive rises?
- There are several other maps similar to the Collatz map and there is also the modified Collatz sequence (where u is replaced with ud), as well as the Syracuse function where long down-steps are collapsed into one down-step. How do these analyses transfer to those contexts?

For the reader who is eager to start exploring Collatz permutations, we have a small code library supporting some basic features for working with their types. The code is written in the Sage open-source mathematics software system, but should run in Python with minor modifications. The code can be found at <https://github.com/SuprDewd/CollatzPermutations>.

3 Acknowledgment.

The authors are grateful to William Stein for access to the Sage Combinat Cluster, supported by NSF Grant No. DMS-0821725.

References

- [1] Jeffrey C. Lagarias, editor. *The ultimate challenge: the $3x + 1$ problem*. American Mathematical Society, Providence, RI, 2010.

This is joint work with Bjarki Gudmundsson and Henning Ulfarsson.

Computing generating functions for the number of descents in permutations which avoid a family of permutations that start with 1.

Quang Bach and Jeffrey Remmel

Department of Mathematics, University of California, San Diego
qtbach@ucsd.edu

Let τ be a permutation in the symmetric group S_p and let $\mathcal{NM}_n(\tau)$ be the set of permutations σ in the symmetric group S_n which have no consecutive occurrences of τ . If $\sigma \in S_n$, we let $\text{des}(\sigma)$ be the number of descents of σ and $\text{LRmin}(\sigma)$ be the number of left-to-right minima of σ . Jones and Remmel [1, 2, 3] showed that if $p \geq 3$ and τ starts with 1, then the generating function

$$M_\tau(x, y, t) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\text{LRmin}(\sigma)} y^{1+\text{des}(\sigma)}$$

is always of the form $\left(\frac{1}{U_\tau(t, y)}\right)^x$ where $U_\tau(t, y) = \sum_{n \geq 0} U_{\tau, n}(y) \frac{t^n}{n!}$. Moreover, they showed that if τ is one of the patterns $1324 \dots p$, $1p2 \dots (p-1)$, or $13 \dots (p-1)2p$ where $p \geq 4$, then the coefficients $U_{\tau, n}(y)$ satisfy simple recursions depending on p . This allowed Jones and Remmel to compute the initial values of $M_\tau(x, y, t)$ for such τ .

The main purpose of this talk is to show that the methods of Jones and Remmel can be extended to apply to permutations which avoid certain families of permutations. That is, if Γ is a set of permutations in S_j , then we let $\mathcal{NM}_n(\Gamma)$ be the set of permutations σ in the symmetric group S_n which have no consecutive occurrences of any of the permutations in Γ . Then if all the permutations in Γ starts with 1, one can show that $M_\Gamma(x, y, t) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} x^{\text{LRmin}(\sigma)} y^{1+\text{des}(\sigma)}$ is always of the form $\left(\frac{1}{\sum_{n \geq 0} U_{\Gamma, n}(y) \frac{t^n}{n!}}\right)^x$. We will discuss various sets Γ where one can show that the coefficient $U_{\Gamma, n}(y)$ satisfy simple recursions. For example, if $p \geq 4$ and $k_1, k_2 \geq 2$ where $k_1 + k_2 = p$, we will show that the coefficients $U_{\Gamma_{k_1, k_2}^p, n}(y)$ satisfy simple recursions where

$$\Gamma_{k_1, k_2}^p = \{\sigma \in S_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, \sigma_1 < \sigma_2 < \dots < \sigma_{k_1} \text{ \& } \sigma_{k_1+1} < \sigma_{k_1+2} < \dots < \sigma_p\}.$$

References

- [1] M. Jones and J.B. Remmel, A reciprocity approach to computing generating functions for permutations with no pattern matches, Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings, 23 International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), **119** (2011), 551-562.
- [2] M. Jones and J. Remmel, A reciprocity method for computing generating function over the set of permutations with no consecutive occurrences of τ , Discrete Mathematics, **313** Issue 23 (2013), 2712-2729.

- [3] M. Jones and J. Remmel, Generating functions for the number of permutations with no consecutive occurrences of $1p23 \dots (p-1)$ or $13(p-1)2p$, to appear in Pure Mathematics and Applications.

On the Möbius function of the quasi-consecutive pattern poset*

Antonio Bernini[†]

Luca Ferrari[†]

Abstract

We define the quasi consecutive pattern poset by declaring $\sigma \leq \tau$ whenever the permutation τ contains an occurrence of the permutation σ in which all the entries are adjacent in τ except at most the first and the second. We then investigate the Möbius function of the quasi consecutive pattern poset and we completely determine it for those intervals $[\sigma, \tau]$ such that σ occurs precisely once in τ .

1 Introduction

Given two permutations σ, τ , we say that $\sigma \leq \tau$ in the permutation pattern poset whenever there is an occurrence of σ in τ as a classical pattern, i.e. there exists a subsequence of (not necessarily consecutive) entries of τ that has the same relative order as σ . An extremely challenging open problem concerning the permutation pattern poset is the determination of its Möbius function. The problem, originally posed by Wilf [W], received quite recently much attention, and some partial results have been achieved [SV, ST, BJJS, Sm]. However, a complete description of the Möbius function of the permutation pattern poset is not yet available. The same problem can be formulated for the *consecutive pattern poset*, where, by definition, $\sigma \leq \tau$ whenever σ appears in τ as a *consecutive* pattern. This poset is much easier than the classical one and, in particular, its Möbius function is now completely understood [BFS, SW].

We recall here that the Möbius function μ of a poset is an important element of the incidence algebra of that poset, i.e. a function mapping an interval of the poset into a scalar. More specifically, it is possible to give a recursive definition of μ as follows:

$$\begin{aligned}\mu(x, x) &= 1 \\ \mu(x, y) &= - \sum_{x \leq z < y} \mu(x, z), \quad \text{if } x \neq y.\end{aligned}$$

In particular, if $x \not\leq y$, then $\mu(x, y) = 0$. Using a duality argument, it is possible to show that the Möbius function of an interval can be equally computed “from top to bottom”, according to the following formula (for $x \neq y$):

$$\mu(x, y) = - \sum_{x < z \leq y} \mu(z, y).$$

Both formulas for computing μ will be frequently used throughout the paper.

Consecutive and classical patterns are special (actually, extremal) cases of the more general notion of *vincular patterns*. An occurrence of a vincular pattern is an occurrence of that pattern in which entries are subject to certain adjacency conditions. Vincular patterns were introduced by Babson and Steingrímsson [BS] (under the name of generalized patterns), and constitute a vast intermediate continent between the two lands of consecutive patterns and classical patterns. Since the determination of the Möbius function is still open in the classical case and totally solved in the consecutive one, it is conceivable that, if one is able to define a reasonable poset structure on permutations depending on the type of vincular patterns under consideration, then the resulting class of permutation posets (which would somehow interpolate between the consecutive and the vincular posets) may shed light on the differences and on the analogies between the two extremal cases.

*Partially supported by PRIN project: “Automi e Linguaggi Formali: Aspetti Matematici e Applicativi”.

[†]Dipartimento di Matematica e Informatica “U. Dini”, viale Morgagni 65, University of Firenze, Firenze, Italy, antonio.bernini@unifi.it, luca.ferrari@unifi.it

In the present work we investigate a single case, probably the closest to the consecutive one, in which all the entries of a pattern are required to be adjacent, except at most the first and the second. We address the problem of the computation of the Möbius function in this poset, and we completely determine the value $\mu(\sigma, \tau)$ when σ occurs precisely once in τ . In spite of its closeness with the consecutive case, it seems that the computation of $\mu(\sigma, \tau)$ in the general case (when more than one occurrence is allowed) is considerably more complicated. This is witnessed, for instance, by the fact (hinted at in the last section) that the absolute value attained by the Möbius function seems to be unbounded, whereas in the consecutive pattern poset the only possible values are $-1, 0, 1$.

2 The quasi-consecutive pattern poset

Given two permutations σ and τ , we say that σ is a *quasi-consecutive pattern* of τ when there is an occurrence of σ in τ whose first two elements are possibly not consecutive, whereas all the remaining elements has to appear consecutively in τ . The resulting poset will be called the *quasi-consecutive pattern poset*. As an example, the permutation 432516 contains the quasi-consecutive pattern 231, since the entries 351 show an occurrence of the vincular pattern $2 - 31$ (in the usual notation for vincular patterns, a dash indicates that two entries need not be adjacent in the bigger permutation, whereas the absence of a dash means that the associated entries are required to be consecutive in the larger permutation). When a permutation τ does not contain the quasi-consecutive pattern σ , we say that τ *avoids* the quasi-consecutive pattern σ . For instance, the preceding permutation 432516 avoids the quasi-consecutive pattern 123. Observe that other instances of vincular patterns of different type having “ground permutation” 123 appear in 432516 (for example, there are occurrences of both $12 - 3$ and $1 - 2 - 3$).

The quasi-consecutive pattern poset has many similarities with the consecutive pattern poset studied in [BFS]. Clearly it has a slightly more complicated structure, which would be interesting to investigate in detail. A first structural property of such a poset is recorded in the next proposition.

Proposition 2.1 *For any $\tau \in S$, τ covers at most three permutations, which are obtained by removing either the first, the second or the last entry of τ (and suitably rearranging the remaining ones).*

The above property gives an important feature of the quasi-consecutive poset, which will be very useful in the proofs of the results stated in the next section.

3 The Möbius function of the quasi-consecutive pattern poset

In this section, which contains the main results of our work, we address the problem of the computation of the Möbius function of the quasi-consecutive pattern poset. Specifically, given $\sigma, \tau \in S$, with $\sigma \leq \tau$, we want to compute $\mu(\sigma, \tau)$. We will completely solve the case of a single occurrence, which turns out to be not a trivial one, and leave the general case as an open problem.

Along this section, we will frequently write permutations with a dash between the first entry and the second entry. This is done in order to emphasize the fact that they have to be interpreted as specific vincular patterns in the quasi-consecutive pattern poset.

We start by giving some partial results, depending on whether τ covers precisely i permutations, for $i = 1, 2$.

Proposition 3.1 *If τ covers precisely one permutation, then $[\sigma, \tau]$ is a chain, for any $\sigma \leq \tau$.*

Thus, if τ covers precisely one permutation, then $\mu(\sigma, \tau) = 1$ when $\sigma = \tau$, $\mu(\sigma, \tau) = -1$ when τ covers σ , and $\mu(\sigma, \tau) = 0$ in all the other cases.

Proposition 3.2 *Let $\tau = a_1 - a_2 \cdots a_n$. If τ covers precisely two permutations, then there are two distinct possibilities:*

- a_1 and a_2 are consecutive integers;
- either $\tau = 1n(n-1) \cdots 32$ or $\tau = n12 \cdots (n-1)$.

Observe that, if τ covers precisely two permutations and a_1 and a_2 are not consecutive integers, then we can determine the structure of the interval $[\sigma, \tau]$, and so its Möbius function. Indeed, if σ is monotone and of length at least 3, then necessarily any occurrence of σ in τ does not involve a_1 . Thus a permutation in $[\sigma, \tau]$ is uniquely determined by a (possibly empty) set of entries to remove from the end of τ and by choosing if a_1 has to be removed or not. For these reasons, it is not difficult to realize that $[\sigma, \tau] \simeq 2 \times n$, where i denotes the chain of the first i positive integers. Therefore, in this case, the computation of the Möbius function can be easily carried out:

$$\mu(\sigma, \tau) = \begin{cases} -1, & \text{if } \sigma \text{ has length } n-1; \\ 1, & \text{if } \sigma \text{ has length } n-2; \\ 0, & \text{otherwise.} \end{cases}$$

On the contrary, if σ is not monotone, then necessarily each occurrence of σ in τ must involve a_1 . In this case, $[\sigma, \tau]$ is a finite chain, whose Möbius function is trivial to compute. Finally, the cases in which the length of σ is at most two are very easy to describe.

3.1 The case of one occurrence

Let $\sigma \leq \tau = a_1 - a_2 \cdots a_n$ and suppose that σ occurs precisely once in τ . We will be concerned with the computation of $\mu(\sigma, \tau)$ in this particular case, which we will be able to solve completely. We will distinguish several cases, depending on the fact that the single occurrence of σ in τ does or does not involve the three elements a_1, a_2 and a_n .

A first obvious consideration is that σ cannot involve all of the three above elements, unless $\sigma = \tau$, and in this last case of course $\mu(\sigma, \tau) = 1$.

The second possibility is that two of the three elements a_1, a_2, a_n are involved in the unique occurrence of σ in τ . There are of course three distinct cases; however in any of them the interval $[\sigma, \tau]$ is easily seen to be a chain. Thus, we can conclude that, in all these cases, $\mu(\sigma, \tau) = 0$, unless σ has length $n-1$, in which case $\mu(\sigma, \tau) = -1$.

We next examine the case in which only one among a_1, a_2 and a_n is involved in the unique occurrence of σ in τ . There are three distinct cases to consider. The first two of them are completely analogous, however we record them in two distinct propositions.

Proposition 3.3 *Suppose that a_1 occurs in σ (whereas a_2 and a_n do not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2 (in which case $\mu(\sigma, \tau) = 1$).*

Proposition 3.4 *Suppose that a_2 occurs in σ (whereas a_1 and a_n do not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2 (in which case $\mu(\sigma, \tau) = 1$).*

The last case is by far the most challenging one, and the proof (which is clearly not reported here to keep the length to a minimum) requires a careful analysis of the structure of $[\sigma, \tau]$.

Proposition 3.5 *Suppose that a_n occurs in σ (whereas a_1 and a_2 do not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2. In this last case, if a_1 and a_2 are consecutive integers, then $\mu(\sigma, \tau) = 0$, otherwise $\mu(\sigma, \tau) = 1$.*

The last case to be considered is when none of a_1, a_2 and a_n belongs to the unique occurrence of σ in τ . It will be convenient to distinguish two cases, depending on whether the occurrence of σ is consecutive or not. In spite of the fact that the two statements are exactly the same, we prefer not to merge them in a single proposition, since the proofs are sensibly different.

Proposition 3.6 *If σ does not occur consecutively in τ , and does not involve a_1, a_2 and a_n , then $\mu(\sigma, \tau) = 0$.*

Proposition 3.7 *If σ occurs consecutively in τ , and does not involve a_1, a_2 and a_n , then $\mu(\sigma, \tau) = 0$.*

4 Further work

The study of the quasi-consecutive pattern poset, as well as of more general vincular pattern posets, which has been initiated in the present work, is of course very far from being completed.

Concerning the main topic investigated here, namely the Möbius function of the quasi-consecutive pattern poset, a lot of work has still to be done. The case of one occurrence which we have completely solved in the previous section seems not to be really representative of the general case. For instance, the absolute value of μ can be different from 0 and 1 (a simple example is given by the interval $[12, 2413]$, whose Möbius function equals 2), and there is computational evidence [St] that $|\mu(\sigma, \tau)|$ is actually unbounded. Despite its closeness with the consecutive case, this fact shows that the quasi-consecutive case can sometimes be very similar to the classical (unrestricted) case. Another conjecture suggested by [St] is the following: if τ is the direct sum of some copies of σ , then $\mu(\sigma, \tau) = 1$.

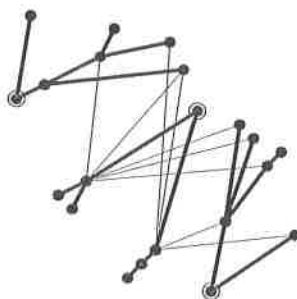
References

- [BS] E. Babson, E. Steingrímsson, *Generalized permutation patterns and a classification of the Mahonian statistics*, Sémin. Lothar. Combin., 44 (2000) Article B44b, 18 pp.
- [BFS] A. Bernini, L. Ferrari, E. Steingrímsson, *The Möbius function of the consecutive pattern poset*, Electron. J. Combin., 18(1) 2011 #P146 (12 pp.).
- [BJJS] A. Burstein, V. Jelínek, E. Jelínková, E. Steingrímsson, *The Möbius function of separable and decomposable permutations*, J. Combin. Theory Ser. A, 118 (2011) 2346–2364.
- [SV] B. E. Sagan, V. Vatter, *The Möbius function of a composition poset*, J. Algebraic Combin., 24 (2006) 117–136.
- [SW] B. Sagan, R. Willenbring, *Discrete Morse theory and the consecutive pattern poset*, J. Algebraic Combin., 36 (2012) 501–514.
- [Sm] J. P. Smith, *On the Möbius function of permutations with one descent*, available at <http://arxiv.org/abs/1306.5926>.
- [ST] E. Steingrímsson, B. E. Tenner, *The Möbius function of the permutation pattern poset*, J. Comb., 1 (2010) 39–52.
- [St] Einar Steingrímsson, *personal communication*.
- [W] H. Wilf, *The patterns of permutations*, Discrete Math., 257 (2002) 575–583.

A fast-growing subset of $\text{Av}(1324)$

David Bevan
The Open University, England

The enumeration of the class of permutations avoiding **1324** is notorious for its difficulty. Even the asymptotic growth rate of the **1324**-avoiders is currently unknown. Recently, Conway & Guttmann have calculated the number of permutations in the class up to length 36, and by analysing the sequence of values give an estimate for the growth rate of $\text{Av}(\mathbf{1324})$ of 11.60 ± 0.01 . However, rigorous bounds still differ from this value quite markedly. In this talk we outline a proof of a new lower bound of 9.81 for the growth rate. (This improves on a previous lower bound of 9.47.)



The Hasse graph of a **1324**-avoider spanned by three trees rooted at alternate boundaries

To each permutation σ , we associate a plane graph H_σ , which we call its *Hasse graph*. The edges of H_σ correspond to the edges of the Hasse diagram of the sub-poset, P_σ , of $\mathbb{N}^2$ consisting of the points (i, σ_i) .

If a permutation avoids the pattern **1324**, then its Hasse graph does not have the diamond graph H_{1324} as a minor. In particular, the subgraph of H_σ induced by a *left-to-right minimum* of σ (a minimal element in P_σ) and the points to its north-east is a *tree*, as is that induced by a *right-to-left maximum* of σ (a maximal element in P_σ) and the points to its south-west.

We consider the set $\mathcal{W}$ of **1324**-avoiders whose Hasse graphs are spanned by a disjoint sequence of trees, rooted at alternate boundaries. The key to our result is an investigation of how these trees can interact. We also need to understand the asymptotic distribution of certain substructures of the Hasse graphs.

Frequently, distributions of parameters counting the proportion of particular substructures in combinatorial classes have a convergent mean and a variance that vanishes asymptotically. Such distributions have the following concentration property:

Proposition. *If ξ_n is a sequence of random variables with means $\mu_n = \mathbb{E}[\xi_n]$ and variances $v_n = \mathbb{V}[\xi_n]$ satisfying the conditions*

$$\lim_{n \rightarrow \infty} \mu_n = \mu, \quad \lim_{n \rightarrow \infty} v_n = 0,$$

for some constant μ , then ξ_n is concentrated at μ in the sense that, for any $\varepsilon > 0$, given sufficiently large n ,

$$\mathbb{P}[|\xi_n - \mu| \leq \varepsilon] > 1 - \varepsilon.$$

In practice this often means that we can work on the assumption that the value of any such parameter is entirely concentrated at its limiting mean. This is the case for the parameters in which we are interested.

Our new lower bound for the growth rate of $\text{Av}(\mathbf{1324})$ follows from an analysis of how the vertices in the sequence of trees spanning elements of $\mathcal{W}$ may be interleaved, proofs that the distributions of the relevant substructures are indeed concentrated, and the determination of the corresponding asymptotic means. As a byproduct, we also prove that the number of occurrences of a fixed pattern in a Łukasiewicz path exhibits a concentrated Gaussian limit distribution.

COMBINATORICS OF SIMPLE MARKED MESH PATTERNS IN 132-AVOIDING PERMUTATIONS

NICOLAS BORIE

ABSTRACT. We present some combinatorial interpretations for coefficients appearing in series partitioning the permutations avoiding 132 along marked mesh patterns. We study patterns in which only one parameter is non zero the combinatorial family in bijection with 132-avoiding permutations and also preserving the statistic counted by the marked mesh pattern. Following the works of Kitaev, Remmel and Tiefenbruck, we present alternative proofs for two quadrants and give a combinatorial interpretation for the last one exhibiting a new bijection between 132-avoiding permutations and non-decreasing parking functions.

1. INTRODUCTION

Mesh patterns were introduced by Brändén and Claesson [BC11] to provide explicit expansions for certain permutation statistics as (possibly infinite) linear combinations of (classical) permutation patterns. This notion was further studied by Úlfarsson [Úlf11] and Kitaev, Remmel and Tiefenbruck in some series of papers refining conditions on permutations and patterns. The present paper focuses on what the trio Kitaev, Remmel and Tiefenbruck call *simple marked pattern in 132-avoiding permutations* [KRT12].

Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation written in one-line notation. We consider the graph $G(\sigma)$ of σ to be the set of points $\{(i, \sigma_i) : 1 \leq i \leq n\}$.

Now, if we draw a coordinate system centered at the point (i, σ_i) , we are interested in the points that lie in the four quadrants I, II, III, IV of that coordinate system as represented on the right of Figure 1. For any a, b, c, d , four non negative integers, we say that σ_i matches the simple marked mesh pattern $MMP(a, b, c, d)$ in σ if, in the coordinate system centered at (i, σ_i) , $G(\sigma)$ has at least a, b, c and d points in the respective quadrants I, II, III and IV.

For $\sigma = 768945213$, then $\sigma_6 = 5$ matches the simple marked mesh pattern $MMP(0, 3, 1, 1)$, since relative to the coordinate system with origin $(6, 5)$, $G(\sigma)$ has respectively 0, 4, 1 and 3 points in Quadrants I, II, III and IV (see Figure 1). If a coordinate in $MMP(a, b, c, d)$ is zero, then there is no condition on the number of points in the corresponding quadrant. We let $mmp(a, b, c, d)(\sigma)$ denote the number of i such that σ_i matches the marked mesh pattern $MMP(a, b, c, d)$ in σ .

Given a sequence $w = w_1 \dots w_n$ of distinct integers, let $pack(w)$ be the permutation obtained by replacing the i -th largest integer that appears in w by i . For example, if $w = 2754$, then $pack(w) = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group $\mathfrak{S}_j$, we say that the pattern τ occurs in a permutation $\sigma \in \mathfrak{S}_n$ if there exist $1 \leq i_1 \leq \dots \leq i_j \leq n$ such that $pack(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$. We say that a permutation σ avoids the pattern τ if τ does not occur in σ . We will denote by $\mathfrak{S}_n(\tau)$ the set of permutations in $\mathfrak{S}_n$ avoiding τ . This paper presents some combinatorial results concerning the generating functions

$$(1) \quad Q_{132}^{(a,b,c,d)}(t, x) := 1 + \sum_{n \geq 1} t^n Q_{n,132}^{(a,b,c,d)}(x)$$

Key words and phrases. permutation statistics, marked mesh pattern, distribution, Catalan numbers, Narayana numbers.

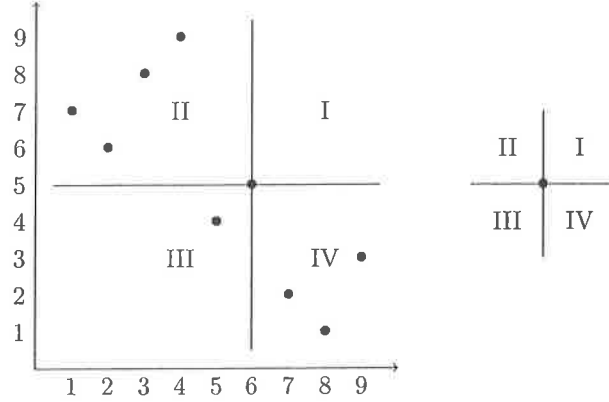


FIGURE 1. The graph of $\sigma = 768945213$ with coordinate system at position 6.

where for any $a, b, c, d \in \mathbb{N}$,

$$(2) \quad Q_{n,132}^{(a,b,c,d)}(x) := \sum_{\sigma \in \mathfrak{S}_n(132)} x^{mmp(a,b,c,d)(\sigma)}.$$

More precisely, we give a combinatorial interpretation of each coefficient of these series when exactly one value among a, b, c, d is non zero. Note that, as explained in [KRT12], $Q_{132}^{(0,0,0,k)}(t, x) = Q_{132}^{(0,k,0,0)}(t, x)$, so that we shall not consider the fourth quadrant.

2. QUADRANT I: THE SERIES $Q_{132}^{(\ell,0,0,0)}$

The marked mesh pattern $MMP(\ell, 0, 0, 0)$ splits the 132-avoiding permutations according to the number of values having at least ℓ greater values on their right.

We recall that a Dyck path with $2n$ steps is a path on the square lattice with steps $(1, 1)$ or $(1, -1)$ from $(0, 0)$ to $(2n, 0)$ that never falls below the x-axis.

Theorem 2.1. $Q_{132}^{(\ell,0,0,0)}|_{t^n x^k}$ counts the number of Dyck paths with $2n$ steps having exactly k steps $(1, -1)$ ending at height greater than or equal to ℓ .

Sketch of proof : The key is the Krattenthaler's bijection [Kra01] between 132-avoiding permutations and Dyck paths.

As the number of increasing steps from height at least 1 in Dyck paths are counted by the Catalan triangle, we deduce the following corollary.

Corollary 2.2. $Q_{132}^{(1,0,0,0)}|_{t^n x^k}$ is equal to the value $C_{n,k}$ of the Catalan triangle.

To illustrate Theorem 2.1, here is an example with $\ell = 2$ and $n = 4$. We have $Q_{4,132}^{(2,0,0,0)}(x) = 8 + 4x + 2x^2$ and Figure 2 shows the fourteen Dyck paths of length 8 with marked decreasing steps ending at height greater than or equal to 2. We have eight Dyck paths under the horizontal line at height 2, four paths containing a single decreasing step ending at height 2, and two last paths containing 2 such decreasing steps.

3. QUADRANT II: THE SERIES $Q_{132}^{(0,\ell,0,0)}$

Quadrant II is about classifying 132-avoiding permutations according to the number of values having ℓ greater values on their left. We will see that this statistic can be read on first values only.

Let n be a positive integer, we will call non-decreasing parking functions of length n the set of all non-decreasing functions f from $1, 2, \dots, n$ into $1, 2, \dots, n$ such that for all $1 \leq i \leq n : f(i) \leq i$. It is well-known that these are enumerated by Catalan numbers.

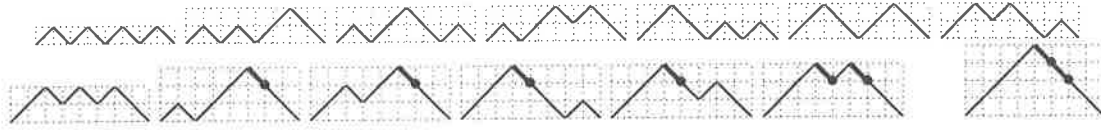


FIGURE 2. Decreasing steps ending at height greater than or equal to 2 in Dyck paths of length 8.

3.1. The general result.

Proposition 3.1. *Let σ be a 132-avoiding permutation. The statistic $mmp(0, 1, 0, 0)$ only depends on the first value of σ . More precisely, $mmp(0, 1, 0, 0)(\sigma) = \sigma_1 - 1$.*

This first proposition is the key to understanding the combinatorics attached to quadrant II. We will also understand it as follows: if a position i matches $MMP(0, 1, 0, 0)$ in σ , thus $\sigma_1 \geq \sigma_i$.

Theorem 3.2. $Q_{132}^{(0, \ell, 0, 0)}|_{t^n x^k}$ is equal to the number of 132-avoiding permutations in $\mathfrak{S}_n$ such that the packed of their suffix of length $n + 1 - \ell$ begins with value $k + 1$.

For example, with $\ell = 2$ and $n = 4$, we have $Q_{4, 132}^{(0, 2, 0, 0)}(x) = 4 + 6x + 4x^2$. The following table displays the fourteen permutations of length 4 avoiding 132, their suffix of length $4 + 1 - 2 = 3$, the packed of the suffix and the statistic $mmp(0, 2, 0, 0)$. The permutation are enumerated by lexicographic order, however the reader can check that four of them contain no value having two greater values on their left, six contain a single such value and the last four contain two positions matching with $MMP(0, 2, 0, 0)$.

σ avoiding 132	suffix of σ of length 3	packed of the suffix	$mmp(0, 2, 0, 0)(\sigma)$
1234	234	123	0
2134	134	123	0
2314	314	213	1
2341	341	231	1
3124	124	123	0
3214	214	213	1
3241	241	231	1
3412	412	312	2
3421	421	321	2
4123	123	123	0
4213	213	213	1
4231	231	231	1
4312	312	312	2
4321	321	321	2

3.2. A new bijection between 132-avoiding permutations and non-decreasing parking functions. Let $\sigma \in \mathfrak{S}_n$ be a 132-avoiding permutation. We set

$$(4) \quad \phi(\sigma) := (mmp(0, n, 0, 0) + 1, mmp(0, n - 1, 0, 0) + 1, \dots, mmp(0, 1, 0, 0) + 1)$$

Theorem 3.3. *Let n be a positive integer. ϕ establishes a bijection between 132-avoiding permutations of length n and non-decreasing parking functions of length n .*

Corollary 3.4. *The number of 132-avoiding permutations containing k values having a greater value on their left is the coefficient $C_{n, k}$ of the Catalan triangle: $Q_{132}^{(0, 1, 0, 0)}|_{t^n x^k} = C_{n, k}$.*

Theorem 3.5. $Q_{n, 132}^{(0, k, 0, 0)}(0)$ is equal to the sum of the k first values of the n^{th} row of the Catalan triangle.

From a combinatorial point of view, it is simpler to present $Q_{n,132}^{(0,k,0,0)}(0)$ as a series in t in columns:

$$(5) \quad \begin{array}{c|cccccccc} n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5 \\ 4 & 1 & 4 & 9 & 14 & 14 & 14 & 14 & 14 \\ 5 & 1 & 5 & 14 & 28 & 42 & 42 & 42 & 42 \\ 6 & 1 & 6 & 20 & 48 & 90 & 132 & 132 & 132 \\ 7 & 1 & 7 & 27 & 75 & 165 & 297 & 429 & 429 \\ 8 & 1 & 8 & 35 & 110 & 275 & 572 & 1001 & 1430 \end{array}$$

We thus recognize a simple recurrence

Theorem 3.6. *We have*

$$(6) \quad Q_{n,132}^{(0,k,0,0)}(0) = \begin{cases} 1 & \text{if } n = 1 \text{ or } k = 1, \\ Q_{n-1,132}^{(0,k,0,0)}(0) + Q_{n,132}^{(0,k-1,0,0)}(0) & \text{if } n \geq k, \\ Q_{n,132}^{(0,k-1,0,0)}(0) & \text{if } n < k. \end{cases}$$

We can prove this property with at least three different approaches: the first one is analytic by noticing that Formula (24) presented by [KRT12] at $x = 0$ imply our relations, the second method builds a bijection between the related sets, the last one shows that simple combinatorics objects counted by these numbers satisfy the induction.

4. QUADRANT III: THE SERIES $Q_{132}^{(0,0,\ell,0)}$

These series refine the 132-avoiding permutations along the number of values having ℓ smaller values on their left.

We recall that the Narayana triangle (A001263 of [Slo03])

$$(7) \quad \begin{array}{c|cccccccc} n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & & & & & & & \\ 2 & 1 & 1 & & & & & & \\ 3 & 1 & 3 & 1 & & & & & \\ 4 & 1 & 6 & 6 & 1 & & & & \\ 5 & 1 & 10 & 20 & 10 & 1 & & & \\ 6 & 1 & 15 & 50 & 50 & 15 & 1 & & \\ 7 & 1 & 21 & 105 & 175 & 105 & 21 & 1 & \\ 8 & 1 & 28 & 196 & 490 & 490 & 196 & 28 & 1 \end{array}$$

We will denote $N(n, k)$ the Narayana number indexed by n and k .

Theorem 4.1. *We have $\forall n \geq 1, \forall 0 \leq k \leq n - 1$,*

$$(8) \quad Q_{132}^{(0,0,1,0)}(t, x)|_{t^n x^k} = N(n, k + 1) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}$$

This theorem precise Theorem 11 of [KRT12].

A way to see that is to consider the binary decreasing tree associated with a 132-avoiding permutation.

Proposition 4.2. *Let σ be a 132-avoiding permutation of $\mathfrak{S}_n$. The number $mmp(0, 0, 1, 0)$ of positions matching with $MMP(0, 0, 1, 0)$ is equal to the number of left branches in the binary decreasing tree associated with σ .*

Proposition 4.3. *The number of binary trees over n nodes containing k left (or right) branches is counted by the Narayana number $N(n, k + 1)$.*

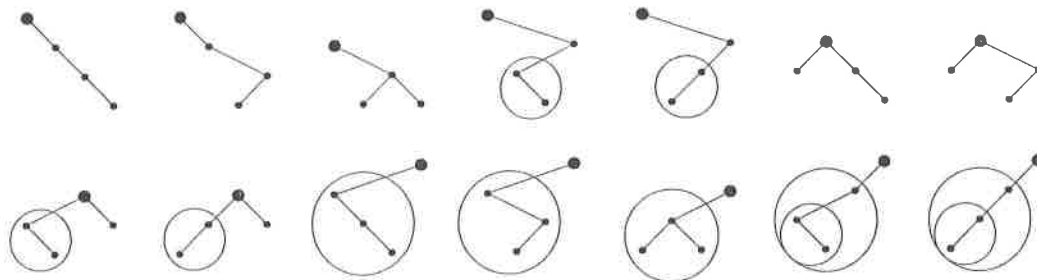


FIGURE 3. Decreasing trees of 132-avoiding permutations of length 4.

Proposition 4.3 and the previous observation conclude Theorem 4.1. We now extend this previous statement from pattern $(0, 0, 1, 0)$ to pattern $(0, 0, \ell, 0)$ with ℓ any positive integer.

Theorem 4.4. $Q_{132}^{(0,0,\ell,0)}|_{t^n x^k}$ is equal to the number of binary trees over n nodes containing themselves k left subtrees containing at least ℓ nodes.

For example, with $\ell = 2$ and $n = 4$, we have $Q_{4,132}^{(0,0,\ell,0)}(x) = 5 + 7x + 2x^2$. As one can check on Figure 3 displaying the fourteen binary trees over 4 nodes in which the left sub-trees over at least 2 nodes have been circled.

ACKNOWLEDGEMENTS

The author would like to thanks Jean-Christophe Novelli for useful discussions and comments. This research was driven by computer exploration using the open-source mathematical software Sage [S⁺09] and its algebraic combinatorics features developed by the Sage-Combinat community [SCc08].

REFERENCES

- [BC11] Petter Brändén and Anders Claesson. Mesh patterns and the expansion of permutation statistics as sums of permutation patterns. *Electron. J. Combin.*, 18(2):Paper 5, 14, 2011.
- [Kra01] C. Krattenthaler. Permutations with restricted patterns and Dyck paths. *Adv. in Appl. Math.*, 27(2-3):510–530, 2001. Special issue in honor of Dominique Foata’s 65th birthday (Philadelphia, PA, 2000).
- [KRT12] Sergey Kitaev, Jeffrey Remmel, and Mark Tiefenbruck. Marked mesh patterns in 132-avoiding permutations i, 2012.
- [S⁺09] W.A. Stein et al. *Sage Mathematics Software (Version 3.3)*. The Sage Development Team, 2009. <http://www.sagemath.org>.
- [SCc08] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008.
- [Slo03] N. J. A. Sloane. The on-line encyclopedia of integer sequences. *Notices Amer. Math. Soc.*, 50(8):912–915, 2003.
- [Úlf11] Henning Úlfarsson. A unification of permutation patterns related to Schubert varieties. *Pure Math. Appl. (P.U.M.A.)*, 22(2):273–296, 2011.

UNIV. PARIS EST MARNE-LA-VALLÉE, LABORATOIRE D’INFORMATIQUE GASPARD MONGE, CITÉ DESCARTES, BÂT COPERNIC – 5, BD DESCARTES CHAMPS SUR MARNE 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE

A general theory of Wilf-equivalence for permutation classes $\text{Av}(231, \pi)$, and other Catalan structures

Mathilde Bouvel, joint work with Michael Albert

The Catalan numbers are renowned for their ubiquity in problems of combinatorial enumeration. A few of the many contexts in which they arise are: plane forests (counted by number of nodes), non-crossing matchings or arch systems (counted by number of matched pairs or arches), Dyck paths, and 231-avoiding permutations. These contexts share the additional property that each admits a natural substructure relation, and that there are bijections between them which preserve that relationship. So, one can further consider those structures of each type which do not contain some designated substructure(s). By contrast, the bijection with complete plane binary trees (counted by number of internal nodes) does not respect the substructure relationship, as observed in [3], which explores another collection of Wilf-equivalences for such structures.

In the talk, we will discuss mostly arch systems and permutations. At *Permutation Patterns*, we will be essentially interested in interpreting our results in terms of permutations. However, for ease of explanation, it will be more convenient to expose our work in the context of arch systems. Of course, there is no essential difference between these two contexts, as it is easy to go bijectively from 231-avoiding permutations to arch systems, while preserving the substructure relation. The bijection is given by $\phi : \sigma = LnR \mapsto \phi(L)[\phi(R)]$, where n is the maximum of σ . An example is shown on Figure 1.



Figure 1: The arch system corresponding to the 231-avoiding permutation 41327658.

As part of a previous work (see an extended abstract [1], or [2]) we considered certain coincidences of enumeration (or *Wilf-equivalences*) between permutation classes $\text{Av}(231, \pi)$, defined by the avoidance of 231 and π , for some 231-avoiding permutation π . Using a non-standard bijection we were able to explain some of those coincidences. However, when we turned to the more general question:

How many distinct enumeration sequences are there for classes of 231-avoiding permutations defined by a single additional restriction?

we were struck by the difference between the computed numbers, and any known general equivalences. Specifically it seemed that there were many more such coincidences (and so fewer enumeration sequences) than one might have expected. We will explain this phenomenon in this talk. Namely, we will show that although there are $\text{Cat}_n = \binom{2n}{n}/(n+1) \sim (1/\sqrt{\pi})n^{-3/2}4^n$ distinct classes of permutations avoiding 231 and an additional permutation of size n , these classes have asymptotically at most $cn^{-3/2}\gamma^n$ distinct enumeration sequences where $c \approx 0.455$ and $\gamma \approx 2.4975$ (these are approximate values only).

A particularly wide collection of such classes share generating functions derived from the continued fraction representation of $C(t) = \sum \text{Cat}_n t^n$, the generating function of the Catalan numbers. Since $C = 1/(1 - tC)$ it follows that:

$$C = \frac{1}{1 - \frac{t}{1 - \frac{t}{1 - \frac{t}{1 - \dots}}}}$$

This fraction can be truncated after n levels, producing a sequence of generating functions:

$$C_0 = 1$$

$$C_n = \frac{1}{1 - tC_{n-1}} \quad \text{for } n \geq 1$$

The functions C_n enumerate many specific subclasses of the Catalan classes above – for instance the 231-avoiding permutations that also avoid a descending permutation of size n . Other examples can be found in [1,4]. Previously these enumeration coincidences were understood on an analytic (or perhaps more properly arithmetic) level only. We can explain them, and many others, bijectively – among other things we can show:

The number of 231-avoiding permutations, π , of size n for which the generating function of the class of permutations avoiding both 231 and π is $C_n(t)$ is the n^{th} Motzkin number.

The proof of this fact also describes (at least in principle) bijections between any two such classes. Furthermore, we show that for any other 231-avoiding permutation θ of size n , the generating function for 231 and θ -avoiding permutations is dominated (term by term and eventually strictly) by $C_n(t)$.

The main tool in producing these results is a binary relation on Catalan structures defined purely intrinsically by four very simple rules.

Definition 1. *The binary relation, $\sim$, defined in the context of arch systems, is the finest equivalence relation that satisfies:*

$$A \sim B \implies \overline{A} \sim \overline{B} \tag{1}$$

$$a \sim b \implies PaQ \sim PbQ \tag{2}$$

$$\top \implies PabQ \sim PbaQ \tag{3}$$

$$\top \implies a\overline{bc} \sim \overline{ab}c. \tag{4}$$

where A , B , P and Q denote arbitrary arch systems; a , b and c denote atoms (i.e. arch systems having a single outermost arch) or empty arch systems; and $\top$ denotes “true”.

This relation induces an equivalence relation – that we also denote $\sim$ – on arch systems (or more generally on Catalan structures) whose equivalence classes are the connected components of the binary relation. The equivalence classes of $\sim$ will be called *cohorts*.

Our main result is the following:

Theorem 2. *If A and B are arch systems and $A \sim B$ then the collection of arch systems not containing A has the same generating function as the collection of arch systems not containing B .*

In other words, the relation $\sim$ refines Wilf-equivalence. We have been able to verify that through size 15 (where $\sim$ has 16,709 equivalence classes on the 9,694,845 Catalan structures) that A and B are $\sim$ -equivalent if and only if the corresponding generating functions are the same. So we have:

Conjecture 3. *The equivalence relation $\sim$ coincides with Wilf-equivalence.*

We will present some elements of the proof of Theorem 2, and discuss Conjecture 3 and further open problems.

References

- [1] Michael Albert and Mathilde Bouvel. Operators of equivalent sorting power and related Wilf-equivalences. *DMTCS Proceedings (FPSAC)* AS (671–682), 2013.
- [2] Michael Albert and Mathilde Bouvel. Operators of equivalent sorting power and related Wilf-equivalences. *ArXiv e-prints*, 2014.
- [3] Michael Dairiko, Lara Pudwell, Samantha Tyner, and Casey Wynn. Non-Contiguous Pattern Avoidance in Binary Trees. *Elec. J. Combin.* 19(3), 2012. Article P22.
- [4] Toufik Mansour and Alek Vainshtein. Restricted permutations and Chebyshev polynomials. *Séminaire Lotharingien de Combinatoire* 47, 2002. Article B47c.

UNBALANCED WILF-EQUIVALENCE

JONATHAN BLOOM, ALEXANDER BURSTEIN*, VÍT JELÍNEK, AND JAY PANTONE

1. INTRODUCTION

A *pattern* is an equivalence class of sequences under order-isomorphism. Two sequences π_1 and π_2 over totally ordered alphabets are order-isomorphic if, for any pair of positions i and j , we have $\pi_1(i) < \pi_1(j)$ if and only if $\pi_2(i) < \pi_2(j)$. We identify a pattern with its canonical representative, in which the k th smallest letter is k . We say that a permutation π *contains* a pattern σ if π has a subsequence order-isomorphic to σ , otherwise we say that π *avoids* σ .

We denote the set of permutations of length n by S_n , the set of permutations in S_n avoiding pattern π by $\text{Av}_n(\pi)$, and the set of permutations in S_n avoiding every pattern in a set Π by $\text{Av}_n(\Pi)$. We say that two (sets of) patterns π' and π'' are *Wilf-equivalent*, denoted $\pi' \sim \pi''$, if $|\text{Av}_n(\pi')| = |\text{Av}_n(\pi'')|$ for all $n \in \mathbb{N}$. We call a Wilf-equivalence *unbalanced* if the two sets of patterns do not contain the same number of patterns of each length. We will sometimes talk about the *type* of an unbalanced Wilf-equivalence, defined by the lengths of the patterns. The Wilf-equivalences we will consider here have types $(4) \sim (4, 7)$ and $(4, 4) \sim (4, 4, 6)$. Furthermore, we present a generalization of the type $(4) \sim (4, 7)$ to type $(n) \sim (n, S)$, where $n \geq 4$ and S is a list of pattern lengths that are between $n + 3$ and $2n - 1$.

2. A $(4) \sim (4, 7)$ WILF-EQUIVALENCE AND GENERALIZATIONS

Our first main result in this section is the following.

Theorem 2.1 ([8, 10]). $\{1324, 3416725\} \sim 1234$.

This result has been proved independently by Burstein and Pantone [8] and (unpublished) in the thesis of Jelínek [10].

We will sketch the principal details of the proof of Theorem 2.1.

A *skew-Ferrers board* is a shape obtained as a difference of two Ferrers boards with a common bottom left corner. A *transversal* (or a *rook placement*) of a skew-Ferrers board B is a subset S of cells of B such that each row and each column of B contain exactly one cell in S . A skew-Ferrers board B is called *permissible* if it admits an antidiagonal transversal (from its top left corner to its bottom right corner).

Consider the skew-Ferrers boards I_2 , J_2 and ρ filled with 0s and 1s and defined as in Figure 1.

Theorem 2.1 follows (corresponding lemmas omitted) from the following result.

Theorem 2.2 ([8, 10]). *Every permissible skew-Ferrers board admits a unique I_2 -avoiding transversal and a unique (J_2, ρ) -avoiding transversal.*

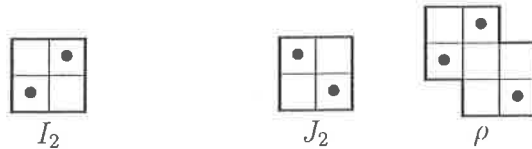


FIGURE 1. Boards I_2 , J_2 , and ρ . Positions of 1s are denoted by the dots. The remaining positions are filled with 0s.

Proof. (SKETCH) The I_2 -avoiding transversal is simply the antidiagonal. Bóna [6] showed that every permissible skew-Ferrers board admits at least one J_2 -avoiding transversal. Given a J_2 -avoiding transversal S of a permissible skew-Ferrers board B , consider the sequence $\mathbf{r}_S = (r_1, \dots, r_{|S|})$ of row numbers of cells in each S (where r_j is the row number from the bottom up of the cell in S in the j th column of B from the left). Then S^* avoids (J_2, ρ) if and only if $\mathbf{r}_{S^*}$ is the lexicographically smallest sequence over all $\mathbf{r}_S$ for J_2 -avoiding transversals S of B (see [10]).

Note that one may also construct S^* recursively (see [8]). Suppose that the bottom row of B has k cells, and the rightmost column of B has l cells. Then put a 1 in the leftmost cell in the bottom row if $k \leq l$ and into the top cell in the rightmost column if $k > l$. Remove the row and column containing this 1 and continue according to the same rule until no cells remain. The cells with 1s yield the desired transversal S^* . $\square$

Since each of I_2 , J_2 and ρ are symmetric about the antidiagonal, we can consider the reversal of those and adapt the proof of Theorem 2.2 to involutions (i.e. permutations π such that $\pi^{-1} = \pi$). Let I_n be the set of all involutions of length n and, for a set of patterns T , let $I_n(T)$ be the set of involutions in I_n avoiding T .

Theorem 2.3. *The set of patterns $(4231, 5276143)$ is Wilf-equivalent to pattern 4321 on involutions. Moreover, for all $n \geq 0$, we have $|I_n(4231, 5276143)| = |I_n(4321)| = M_n$, the n th Motzkin number.*

Given permutations α and β , let $\alpha \oplus \beta$ be the permutation of length $|\alpha| + |\beta|$, whose prefix A of length $|\alpha|$ is an instance of pattern α , suffix B of length $|\beta|$ is an instance of pattern β , and every entry in A is less than every entry in B (e.g. $1324 = 1 \oplus 21 \oplus 1$).

Then Theorem 2.1 can be generalized as follows.

Theorem 2.4. *For any permutations $\sigma, \tau \neq \emptyset$, we have $\sigma \oplus 12 \oplus \tau \sim \{\sigma \oplus 21 \oplus \tau, R\}$, where R is a certain set of patterns of lengths from $|\sigma| + |\tau| + 5$ to $2|\sigma| + 2|\tau| + 3$ related to ρ .*

3. $A(4, 4) \sim (4, 4, 6)$ WILF-EQUIVALENCE

EGGE [9] conjectured that permutations avoiding the set of patterns $\{2143, 3142, \tau\}$, where $\tau \in \{246135, 254613, 263514, 524361, 546132\}$, are enumerated by the large Schröder numbers (and thus $\{2143, 3142, \tau\}$ with τ as above is Wilf-equivalent, for example, to the set of patterns $\{2413, 3142\}$, among other pairs of 4-letter patterns).

Our main result here is the proof of all conjectures of Egge.

Theorem 3.1 ([5, 8]). *$\{2143, 3142, \tau\} \sim \{2413, 3142\}$, where τ is any pattern among $246135, 254613, 263514, 524361, 546132$.*

We omit the proof in this abstract. Burstein and Pantone [8] proved the case of $\tau = 246135$. Bloom and Burstein [5] proved the remaining four cases. We only note here that in some of the cases, we begin by enumerating the simple permutations in the class $\text{Av}(2143, 3142, \tau)$.

A permutation π contains an *interval* if for some $i \leq j$ and $k \leq l$ with $j - i = l - k$, we have $\pi([i, j]) = [k, l]$. A permutation π is called *simple* if it only contains intervals of length 1 and $|\pi|$ (i.e. no nontrivial intervals). For a set S of permutations, let $\text{Si}(S)$ be the set of simple permutations in S . Then we have the following results.

Lemma 3.2 ([8]). $\text{Si}(\text{Av}(2143, 3142, 246135)) = \text{Si}(\text{Av}(2143, 3142, 3412))$.

Lemma 3.3 ([5]). $\text{Si}(\text{Av}(2143, 3142, 263514)) = \text{Si}(\text{Av}(2143, 3142, 4132))$.

4. FUTURE DIRECTIONS

Computer search suggests many unbalanced Wilf-equivalences of the types $(4, 4) \sim (4, 4, 5)$, $(4, 4) \sim (4, 4, 6)$, etc., some of which have been proved. The list is too long to include here, but we would like to note a conjectured Wilf-equivalence of the type $(4) \sim (4, 6)$.

Conjecture 4.1 ([8]). $3142 \sim \{2143, 415263\}$.

The pair $3142 \sim \{2143, 415263\}$ appears preferable to their inverses $2413 \sim \{2143, 246135\}$ mentioned in [8], since $\text{Av}(3142)$ and $\text{Av}(2143, 415263)$ appear to have the same distribution of both the left-to-right minima positions and the descent set (and hence the major index). Thus, it would be nice to find a bijection between the two sets above that preserves those statistics.

Since $|\text{Av}_5(\pi)| = 103$ for all π with $|\pi| = 4$, there can be no $(4) \sim (4, 5)$ Wilf-equivalence. Computation has verified that, up to symmetries, there are no unbalanced Wilf-equivalences of the form $(4) \sim (4, 6)$ or $(4) \sim (4, 7)$ other than those in Theorem 2.1 and Conjecture 4.1. It follows that there are no other $(4) \sim (4, k)$ Wilf-equivalences for any k .

REFERENCES

- [1] M.H. ALBERT, PermLab: Software for permutation patterns. Available online at <http://www.cs.otago.ac.nz/PermLab>, 2012.
- [2] M.H. ALBERT, M.D. ATKINSON, Simple permutations and pattern restricted permutations, *Discrete Math.* **300** (2005), no. 1-3, 1-15.
- [3] M.H. ALBERT, V. VATTER, Generating and enumerating 321-avoiding and skew-merged simple permutations. *Electron. J. Combin.* **20(2)** (2013), Paper 44, 11 pp.
- [4] M.D. ATKINSON, M.M. MURPHY, N. RUŠKUC, Sorting with two ordered stacks in series, *Theoret. Comput. Sci.* **289** (2002), 205-223.
- [5] J. BLOOM, A. BURSTEIN, Egge triples and unbalanced Wilf-equivalence, in preparation.
- [6] M. BÓNA, Permutations avoiding certain patterns: The case of length 4 and some generalizations, *Discrete Math.* **175** (1997), no. 1-3, 55-67.
- [7] M. BÓNA, C. HOMBERGER, J. PANTONE, V. VATTER, Pattern-avoiding involutions: exact and asymptotic enumeration, [arXiv:1310.7003](https://arxiv.org/abs/1310.7003).
- [8] A. BURSTEIN, J. PANTONE, Two examples of unbalanced Wilf-equivalence, [arXiv:1402.3842](https://arxiv.org/abs/1402.3842).
- [9] E. EGGE, Some new pattern-avoiding permutations counted by the Schröder numbers, talk, AMS Fall Eastern Meeting, Rochester, NY, September 2012.
- [10] V. JELÍNEK, Wilf-type classifications: Extremal and enumerative theory of ordered structures, Ph.D. thesis, Charles University, Prague, 2008.
- [11] C. KRATTENTHALER, Permutations with restricted patterns and Dyck paths, *Adv. Appl. Math.* **27** (2001), no. 2-3, 510-530.

- [12] R. SIMION, F.W. SCHMIDT, Restricted permutations, *Europ. J. Combin.* **6** (1985), no. 4, 383–406.
- [13] Z.E. STANKOVA, Forbidden subsequences, *Discrete Math.* **132** (1994), no. 1-3, 291–316.

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755-3551
E-mail address: jonathan.bloom@dartmouth.edu
URL: <http://www.math.dartmouth.edu/~jbloom>

DEPARTMENT OF MATHEMATICS, HOWARD UNIVERSITY, WASHINGTON, DC 20059
E-mail address: aburstein@howard.edu
URL: <http://www.alexanderburstein.org>

COMPUTER SCIENCE INSTITUTE, CHARLES UNIVERSITY, MALOSTRANSKÉ NÁMĚSTÍ 25, PRAGUE 1,
118 00, CZECH REPUBLIC
E-mail address: jelinek@iuuk.mff.cuni.cz
URL: <http://kam.mff.cuni.cz/~jelinek>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FL 32601
E-mail address: jaypantone@ufl.edu
URL: <http://www.jaypantone.com>

INVERSION POLYNOMIALS FOR PERMUTATIONS AVOIDING CONSECUTIVE PATTERNS

KENDRA KILLPATRICK AND NAIOMI T. CAMERON*

ABSTRACT. We use the language of Fibonacci tableaux to study the inversion polynomial for permutations that avoid Π where Π is a collection of consecutive patterns of length three.

1. INTRODUCTION

In 2012, Sagan and Savage introduced the notion of st -Wilf equivalence for a statistic st and for sets of permutations that avoid particular permutation patterns [14]. In this paper, we consider inv -Wilf equivalence on sets of two or more consecutive permutation patterns.

The study of permutations that avoid a given pattern has been a topic of much interest in the past decade. Early work on generalized permutation patterns was done by Elizalde and Noy [10] and Kitaev [12] in 2003. Elizalde and Noy gave the first systematic study of permutations that avoid generalized permutation patterns of length 3, primarily focusing on consecutive patterns of length 3. Kitaev considered simultaneous avoidance of two or more consecutive patterns of length 3, providing results for the number of permutations that avoid any subset of four such consecutive patterns and for all but one subset of three such consecutive patterns. Kitaev also gave several results for subsets of size two. Kitaev and Mansour [13] continued this work, enumerating the permutations that avoid the remaining size three subset, and Aldred, Atkinson and McCaughan [1] completed the enumeration of the permutations that avoid the subsets of size two. In 2012, Elizalde and Noy [9] studied pattern avoidance for subsets of consecutive patterns of length greater than three.

In 2013, Cameron and Killpatrick gave the inversion generating function for Fibonacci tableaux which are in one-to-one correspondence with the set of permutations that simultaneously avoid the consecutive patterns 321 and 312 [3]. In the present paper, we use Fibonacci tableaux and a more general notion of strip tableaux to prove recursive formulas for inversion polynomials on other sets.

2. PRELIMINARIES

2.1. Consecutive pattern avoidance. For any $\pi \in S_n$ and $\sigma \in S_k$, we say that π contains a copy of the pattern σ if π has a subsequence that is order isomorphic to σ . In this case, we refer to σ as a pattern of length k . If π contains no subsequence order isomorphic to σ then we say that π avoids σ . Generalized permutation patterns

2000 *Mathematics Subject Classification.* 05A19, 11B39.

Key words and phrases. tableaux, Fibonacci tableaux, consecutive patterns, permutation statistics, inversions, generalized pattern avoidance, Wilf equivalence.

were first introduced by Babson and Steingrímsson in 2000 [2]. In a *generalized permutation pattern* σ , we may require some of the letters in σ to be adjacent. We place a dash between elements of the pattern whenever these elements need not be adjacent in Π and place no dash otherwise. For example, a $3-1-2$ pattern in π would be any subsequence π_i, π_j, π_k for $i < j < k$ with $\pi_j < \pi_k < \pi_i$, while a $31-2$ pattern in π would be any subsequence π_i, π_{i+1}, π_j for $i+1 < j$ with $\pi_{i+1} < \pi_j < \pi_i$. We will refer to a pattern of the sort $\sigma_1 - \sigma_2 - \sigma_3 - \cdots - \sigma_k$ as a *standard pattern* of length k , while a pattern with no internal dashes will be referred to as a *consecutive pattern*.

2.2. st-Wilf Equivalence. Let Π be a set of standard permutation patterns of length k where $k \leq n$ and define $\text{Av}_n(\Pi)$ as the set of permutations in S_n which avoid every pattern in Π . Two sets of permutation patterns Π and Π' are said to be *Wilf equivalent* if $|\text{Av}_n(\Pi)| = |\text{Av}_n(\Pi')|$. If Π and Π' are Wilf equivalent, we write $\Pi \equiv \Pi'$. We can extend this notion to the case where Π and Π' are sets of generalized permutation patterns. As above, we will say Π and Π' are *Wilf equivalent* if $|\text{Av}_n(\Pi)| = |\text{Av}_n(\Pi')|$.

Sagan and Savage [14] recently defined a q -analogue of Wilf equivalence by considering any permutation statistic st from $\mathfrak{S}_{n \geq 0} \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, and letting

$$F_n^{\text{st}}(\Pi; q) = \sum_{\sigma \in \text{Av}_n(\Pi)} q^{\text{st}(\sigma)}.$$

For Π and Π' subsets of permutations, they defined Π and Π' to be *st-Wilf equivalent* if $F_n^{\text{st}}(\Pi; q) = F_n^{\text{st}}(\Pi'; q)$ for all $n \geq 0$. In this case, we write $\Pi \stackrel{\text{st}}{\equiv} \Pi'$. While Sagan and Savage defined this notion for Π and Π' subsets of standard permutation patterns, we can use the same definition for Π and Π' sets of generalized permutation patterns.

2.3. Inversion statistic. Given a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ where $\sigma_i = \sigma(i)$, we define an *inversion* to be a pair (i, j) such that $i < j$ and $\sigma_i > \sigma_j$. Then the *inversion statistic*, $\text{inv}(\sigma)$, is the total number of inversions in σ .

We use $I_n(\Pi; q)$ to denote the inversion polynomial on the set of permutations that avoid the set Π of generalized permutation patterns. That is,

$$I_n(\Pi; q) = F_n^{\text{inv}}(\Pi; q) = \sum_{\sigma \in \text{Av}_n(\Pi)} q^{\text{inv}(\sigma)}$$

2.4. Fibonacci tableaux. Fibonacci tableaux arise from the Fibonacci lattice first defined by Stanley [15] in 1975. We define a *Fibonacci shape* to be a sequence of 1's and 2's. The Ferrers diagram for a Fibonacci shape is formed by replacing a 1 with a single dot and a 2 with two dots. The size of the Fibonacci shape is the sum of the 1's and 2's that make up the shape. For example, the Ferrers diagram for the Fibonacci shape $\mu = 122121$ looks like

$$\begin{array}{ccccccc} & & \bullet & & \bullet & & \bullet \\ & & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

and has size 9.

Given a Fibonacci shape μ of size n and its corresponding Ferrers diagram, a *standard Fibonacci tableau* is a filling of the Ferrers diagram with the integers 1 through n such that the bottom row decreases from left to right and every column

decreases from bottom to top. An example of a standard Fibonacci tableau of shape $\mu = 122121$ is

$$\begin{array}{cccccc} & 3 & 4 & & 2 & \\ 9 & 8 & 7 & 6 & 5 & 1 \end{array}$$

We define the *column-reading word*, $w_c(T)$, for a standard Fibonacci tableau T by reading the columns from bottom to top, right to left. For the tableau above, $w_c(T) = 152674839$. We define $\text{inv}(T)$ for a Fibonacci tableau T to be $\text{inv}(w_c(T))$.

We obtain an immediate bijection between the set of column reading words for standard Fibonacci tableaux and the set of permutations which avoid both the generalized patterns 312 and 321. In [3], the authors used standard Fibonacci tableaux to prove the following recurrence for the inversion polynomial on permutations that avoid $\Pi = \{321, 312\}$:

$$I_n(\Pi; q) = I_{n-1}(\Pi; q) + (q + q^2 + \cdots + q^{n-1})I_{n-2}(\Pi; q).$$

Extending the notion of a Fibonacci tableau to a more general *strip tableau* allows us to model more sets of generalized pattern avoiding permutations.

3. SUMMARY OF RESULTS

In this paper, we prove recursive formulas for all but one case of $I_n(\Pi; q)$ where Π contains three or more consecutive patterns of length three. In the interest of space, some of the results are summarized in Table 1. To illustrate the method of proof, we provide here a proof of just one of the results, found in Class 7 of Table 1. Note that $I_n^k(\Pi; q)$ denotes the inversion polynomial for permutations that avoid Π and end with a decreasing sequence of length k , $\widetilde{I}_n^k(\Pi; q)$ denotes the inversion polynomial for permutations that avoid Π and begin with a decreasing sequence of length k , and $C_n(q)$ denotes the q -Catalan polynomial defined as $C_n(q) = \sum_{k=1}^n q^{k-1} C_{k-1}(q) C_{n-k}(q)$ in [4].

Proof of Class 7. In this case, since the permutations in $\text{Av}_n(\Pi)$ must avoid 321, 312, and 231, this set of permutations is in one-to-one correspondence with standard Fibonacci tableaux for which the elements in the top of each column must decrease from left to right. (Note: the element in a column of height one is both in the top row of its column and in the bottom row of its column.)

If the Fibonacci tableau begins with a column of height one, then there must be an n in this column and n appears as the last element in the word of the tableau. Removing this n will not change the inversion statistic for this permutation and will give a tableau of size $n - 1$ with the given restrictions. Therefore, $I_n^1(\Pi; q) = I_{n-1}(\Pi; q)$.

If the Fibonacci tableau begins with a column of height two, let k be the smallest integer such that the first k columns have height two and the $k + 1$ st column has height one. Since the elements in the top row of each column and the bottom row of each column must decrease from left to right, the numbers n through $n - 2(k) + 1$ must be in the first k columns and $n - 2(k)$ must be in column $k + 1$ of height 1. Thus the tableau formed from the remaining columns to the right of column $k + 1$ corresponds to a permutation of size $n - 2(k) - 1 = n - 2k - 1$. Since $n - 2k$ is in column $k + 1$, removal of this element does not change the inversion statistic and also gives a tableau in the first k columns that corresponds (with relabeling) to a tableau of size $2k$ with all columns of height two whose elements in the top row of each column decrease from left to right. These tableaux are counted by the Catalan

number C_k . This gives us the inversion polynomial $q^k C_k(q) I_{n-2k-1}(\Pi; q)$, and we have $I_n^2(\Pi; q) = \sum_{k=1}^{\lfloor n/2 \rfloor} q^k C_k(q) \cdot I_{n-2k-1}(\Pi; q)$. Thus, with initial conditions $I_2^1(\Pi; q) = 1$ and $I_2^2(\Pi; q) = I_3^2(\Pi; q) = q$, we have

$$I_n(\Pi; q) = I_{n-1}(\Pi; q) + \sum_{k=1}^{\lfloor n/2 \rfloor} q^k C_k(q) \cdot I_{n-2k-1}(\Pi; q).$$

Table 1: Inversion polynomials for permutations avoiding the set Π of consecutive patterns.

Class	Π	$I_n(\Pi; q)$
1	321, 312, 132, 123	$\begin{cases} q^{k(k+1)} + q^{k^2}, & \text{if } n = 2k + 1 \\ q^{k(k-1)} + q^{k^2}, & \text{if } n = 2k \end{cases}$
	321, 231, 213, 123	$\begin{cases} q^{\binom{n}{2} - k(k+1)} + q^{\binom{n}{2} - k^2}, & \text{if } n = 2k + 1 \\ q^{\binom{n}{2} - k(k-1)} + q^{\binom{n}{2} - k^2}, & \text{if } n = 2k \end{cases}$
2	321, 312, 231, 132	$1 + q$
	321, 312, 231, 213	
	312, 213, 132, 123	$q^{\binom{n}{2}} + q^{\binom{n}{2} - 1}$
	231, 213, 132, 123	
3	312, 213, 231, 132	$\begin{cases} 1 + q^{\binom{n}{2}}, & n = 3 \\ 0, & n > 3 \end{cases}$
4	321, 312, 213, 123	$q + q^2$
	321, 231, 132, 123	$q^{\binom{n}{2} - 1} + q^{\binom{n}{2} - 2}$
5	321, 312, 213, 132	$1 + q^2 + \dots + q^{n-1}$
	321, 231, 213, 132	
	312, 231, 213, 123	$q^{\binom{n}{2}} + q^{\binom{n}{2} - 2} + \dots + q^{\binom{n}{2} - n + 1}$
	312, 231, 132, 123	
6	321, 312, 231, 123	$\begin{cases} 2q^k C_k(q) & \text{if } n = 2k + 1 \\ q^{k-1} C_{k-1}(q) + q^k C_k(q) & \text{if } n = 2k \end{cases}$
	321, 213, 132, 123	$\begin{cases} 2q^{\binom{n}{2} - k} C_k(1/q) & \text{if } n = 2k + 1 \\ q^{\binom{n}{2} - k + 1} C_{k-1}(1/q) + q^{\binom{n}{2} - k} C_k(1/q) & \text{if } n = 2k \end{cases}$
7	321, 312, 231	$I_{n-1}(\Pi; q) + \sum_{k=1}^{\lfloor n/2 \rfloor} q^k C_k(q) \cdot I_{n-2k-1}(\Pi; q)$
	213, 132, 123	$q^{\binom{n}{2}} I_{n-1}(\Pi; 1/q) + \sum_{k=1}^{\lfloor n/2 \rfloor} q^{\binom{n}{2} - k} C_k(1/q) \cdot I_{n-2k-1}(\Pi; 1/q)$
8	321, 312, 213	$1 + q + \dots + q^{n-1}$
	321, 231, 132	
	312, 213, 123	$q^{\binom{n}{2}} + q^{\binom{n}{2} - 1} + \dots + q^{\binom{n}{2} - n + 1}$
	231, 132, 123	

Continued on next page

Table 1 – continued from previous page

Class	Π	$I_n(\Pi; q)$
9	312, 231, 132	$1 + q + q^{\binom{n}{2}} + \sum_{k=3}^{n-1} q^{k-1} \widetilde{I_{n-1}^{k-1}}(\Pi; q) + q^{k-2} \widetilde{I_{n-1}^k}(\Pi; q)$
	312, 231, 213	
	231, 213, 132	$1 + q^{\binom{n}{2}-1} + q^{\binom{n}{2}} + \sum_{k=3}^{n-1} q^{\binom{n}{2}-k+1} \widetilde{I_{n-1}^{k-1}}(\Pi; 1/q) + q^{\binom{n}{2}-k+2} \widetilde{I_{n-1}^k}(\Pi; 1/q)$
	312, 213, 132	

REFERENCES

- [1] R. Aldred, M. Atkinson, D. McCaughan, Avoiding consecutive patterns in permutations, *Adv. in Appl. Math.* 45 (2010), no. 3, 449-461.
- [2] E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics. *Sem. Lothar. Combin.* 44 (2000), Art. B44b, 18 pp. (electronic).
- [3] N. Cameron, K. Killpatrick, Symmetry and log-concavity results for statistics on Fibonacci tableaux, *Ann. Comb.* 17 (2013), 603-618.
- [4] Carlitz, L. and Riordan, J. Two element lattice permutation numbers and their q -generalization. *Duke J. Math* 31 (1964), 371-388.
- [5] S. Cheng, S. Elizalde, A. Kasraoui, B. Sagan, Inversion polynomials for 321-avoiding permutations, *Discrete Math.* 313 (2013), no. 22, 2552-2565.
- [6] A. Claesson, Generalized Pattern Avoidance, *European J. Combin.* 22 (2001), no. 7, 961-971.
- [7] T. Dokos, T. Dwyer, B. Johnson, B. Sagan, K. Selsor, Permutation patterns and statistics, *Discrete Math.* 312 (2012), no. 18, 2760-2775.
- [8] S. Elizalde, Asymptotic enumeration of permutations avoiding generalized patterns, *Advances in Applied Mathematics* 36 (2006), 138-155.
- [9] S. Elizalde, M. Noy, Clusters, generating functions and asymptotics for consecutive patterns in permutations, *Adv. in Appl. Math.* 49 (2012), 351-374.
- [10] S. Elizalde, M. Noy, Consecutive Patterns in Permutations, *Adv. in Appl. Math.* 30 (2003), 110-125.
- [11] Killpatrick, K., On the parity of certain coefficients for a q -analogue of the Catalan numbers, *Electron. J. Combin.* 16 (2009), #R00.
- [12] S. Kitaev, Multi-avoidance of generalised patterns. *Discrete Math.* 260 (2003), no. 1-3, 89100.
- [13] S. Kitaev, T. Mansour, Simultaneous avoidance of generalized patterns, *Ars Combinatoria* 75 (2005), 267-288.
- [14] B. Sagan, C. Savage, Mahonian pairs, *J. Combin. Theory Ser. A* 119 (2012), no. 3, 526-545.
- [15] R. Stanley, The Fibonacci Lattice, *Fibonacci Quart.* 13 (1975), 215-232.

PEPPERDINE UNIVERSITY, 24255 PACIFIC COAST HWY, MALIBU, CA 90263-4321
 E-mail address: Kendra.Killpatrick@pepperdine.edu

LEWIS AND CLARK COLLEGE, 0615 SW PALATINE HILL ROAD, PORTLAND, OR 97219
 E-mail address: ncameron@lclark.edu

An infinite family of inv-Wilf-equivalent permutation pairs

Justin H.C. Chan

Abstract

Wilf-equivalence is one of the central concepts of pattern-avoiding permutations, and has been studied for more than thirty years. The two known infinite families of Wilf-equivalent permutation pairs, due to Stankova-West and Backelin-West-Xin, both satisfy the stronger condition of shape-Wilf-equivalence. Dokos *et al.* recently studied a different strengthening of Wilf-equivalence called inv-Wilf-equivalence, which takes account of the inversion number of a permutation. They conjectured that all inv-Wilf-equivalent permutation pairs arise from trivial symmetries. We disprove this conjecture by constructing an infinite family of counterexamples derived from the permutation pair (231) and (312). The key to this construction is to generalize simultaneously the concepts of shape-Wilf-equivalence and inv-Wilf-equivalence. A further consequence is a proof of the recent Baxter-Jaggard conjecture on even-shape-Wilf-equivalent permutation pairs.

The Complexity of Counting Poset and Permutation Patterns

Joshua Cooper* and Anna Kirkpatrick

Mathematics Department, University of South Carolina
1523 Greene St., Columbia SC 29208

We introduce a new notion of pattern matching that generalizes classical permutation patterns as well as poset containment. In the most general formulation, let P and Q be posets which we term the “pattern” and “text” posets, respectively. We say that P “matches” a subposet Q' of Q if there exists an onto function $f : P \rightarrow Q'$ so that f is order-preserving, i.e., $v \prec_P w$ implies $f(v) \prec_{Q'} f(w)$; in this case, f is called a “match” from P to Q . Furthermore, we say that the match is “induced” if f is in fact an order isomorphism, i.e., f is a match so that $f(v) \prec_{Q'} f(w)$ implies $v \prec_P w$, and we say that the match is “injective” (“bijective”) if f is injective (respectively, bijective).

First, we show that several classical problems can be reformated as questions about poset pattern matching. To do so, we require the notion of “poset dimension”. Given a poset P , a “linear extension” of P is a bijective match from P to a chain C , i.e., a total ordering. A family $\mathcal{R} = \{f_1, \dots, f_r\}$ of linear extensions of P is said to be a “realizer” of P if a relation $v \prec w$ is in P iff $f_j(v) \prec_C f_j(w)$ for every $j \in [r]$; a realizer $\mathcal{R}$ is “minimal” iff it has the fewest possible number of elements among all realizers of P ; the cardinality of a minimal realizer of P is the dimension $\dim(P)$ of P . We also require the complexity-theoretic concepts of a decision or functional computational problem being “polynomial time” (roughly, “easy”), “polynomial time reducible to” (roughly, “no harder than”), “NP” (roughly, “easily checkable”), “NP-hard” (roughly, “NP and hard”), “#P-hard” (roughly, “hard to compute as a function”). (The sense in which these definitions are “rough” is not just in their lack of specificity, but also in the assumption that $P \neq NP$.) The “width” of a poset is the size of its largest antichain, i.e., subset of vertices

between which there are no relations. The following are our main results.

1. If $\dim(P) = \dim(Q) = 2$, the problem of computing the number of induced, injective matches from P to Q is $\#P$ -hard and essentially equivalent to classical permutation pattern counting. This follows from the folklore identification of dimension 2 posets with permutations; the $\#P$ -hardness of classical permutation pattern counting due to Bose-Buss-Lubiw; the fact that counting the number of automorphisms of a fixed-dimension poset is polynomially-time reducible to counting the number of realizers; the polynomial-time computability of the number of realizers of a dimension 2 poset from the Gallai modular decomposition by Ille-Rampon; and the polynomial-time computability of the Gallai decomposition by Habib-Paul. Note the contrast with poset automorphism counting in general, which is equivalent to the complexity of graph isomorphism counting, which is, by a result of Mathon, polynomial-time reducible to the graph isomorphism decision problem (and is therefore so-called “GI-complete”). Also worth noting is that it follows that many natural generalizations are $\#P$ -hard, such as counting the number of induced matches between two arbitrary posets.
2. If $\dim(P) = \dim(Q) = 2$, the problem of computing the number of (not necessarily induced) injective matches from P to Q is $\#P$ -hard by a modification of the argument of Bose-Buss-Lubiw. Because of the correspondence with permutations, we can speak of the essentially equivalent problem of counting “not necessarily induced” pattern matches between permutations, where increasing – but not decreasing -- relations are required to be preserved by the map. This perspective also introduces a notion of automorphisms of individual permutations: automorphisms of the corresponding two-dimensional posets, or maps from the ground set to itself that preserve coinversions but not necessarily inversions. (For a permutation $\sigma \in \mathfrak{S}_n$ and $i, j \in [n]$, the pair $\{i, j\}$ is said to be an “inversion” if $(i - j)(\sigma(i) - \sigma(j)) < 0$ and a “coinversion” if $(i - j)(\sigma(i) - \sigma(j)) > 0$.)
3. If $\dim(Q) = 1$, the problem of counting the number of (injective or not necessarily injective) matches from an arbitrary P to Q is $\#P$ -hard by Brightwell-Winkler’s famous result that counting the number of linear extensions of a poset is $\#P$ -hard. By contrast, some special cases of this problem are in fact easy. Define the “intrinsic width” of a poset

is defined to be the maximum width of the indecomposable nodes in a Gallai modular decomposition. (So, for example, series-parallel posets are characterized by having intrinsic width 1.) If the intrinsic width of a poset is bounded, by results of Steiner and others, the number of linear extensions of P can be computed in polynomial time, whence the number of matches into a chain can be computed in polynomial time.

4. If $\dim(P) = 1$, then counting the number of matches into an arbitrary poset Q is equivalent to counting the number of independent sets in the complement of the comparability graph of Q , shown to be $\#P$ -hard by Provan-Ball as a simple reduction from the classical result of Valiant on maximal independent sets in a graph.

A plethora of questions remain open about poset pattern matching. For example, if $\dim(P) = 2$ and Q is a chain, is the problem of counting the number of (not necessarily induced, injective) matches of P into Q $\#P$ -hard? This is equivalent to the question of whether counting linear extensions of dimension 2 posets is hard, a question we show (using Brightwell-Kostochka-Trotter-Winkler's result about the dimension of layers of the Boolean lattice) is left open by Brightwell-Winkler, since their construction involves posets whose dimensions tend to infinity. What are the complexity of poset pattern recognition problems? What are the complexity of the related questions where injectivity is not required? What problems/results from the substantial literature on permutation pattern avoidance generalize in an interesting way to not necessarily induced matches?

Connections between rook monoid pattern avoidance and other combinatorial objects

Dan Daly

For any natural number n , the rook monoid $\mathcal{R}_n$ is the set of all $0-1$ $n \times n$ matrices such that every row and column contains at most one 1. One can view an element $\pi \in \mathcal{R}_n$ as a string $\pi = \pi_1 \dots \pi_n$ with the properties that for all $1 \leq i \leq n$, $\pi_i \in \{0, 1, \dots, n\}$ and if $\pi_i = \pi_j > 0$ then $i = j$. Rook monoids naturally generalize symmetric groups and as such pattern avoidance generalizes to rook monoids as well. We define avoidance in rook monoids as:

Definition 1. Let $\rho \in \mathcal{R}_m$ and $\pi \in \mathcal{R}_n$. We say that π contains ρ as a (one-dimensional) pattern if there exist $1 \leq i_1 < i_2 < \dots < i_m \leq n$ such that $\pi_{i_\ell} = 0$ if and only if $\rho_\ell = 0$ and for $\pi_{i_a}, \pi_{i_b} > 0$, $\pi_{i_a} > \pi_{i_b}$ if and only if $\rho_a > \rho_b$. If π does not contain ρ , then we say that π avoids ρ .

For example, 405021 contains 102 and 201 as patterns, but not 1002.

We can then define $r_n(\pi)$ to be the set of all elements of $\mathcal{R}_n$ that avoid π . In studying the enumerations of $r_n(\pi)$ for various π , some interesting connections with other combinatorial objects arise. We will explore how rook monoid avoidance connects with A-reduced elements in Coxeter groups of type B as defined by Stembridge [4] and reverse-complement invariant permutations as defined by Egge [3].

References

- [1] A. Bjorner and F. Brenti, *Combinatorics of Coxeter Groups*, Springer, New York, NY (2005).
- [2] M. B. Can and L. E. Renner, The Bruhat-Chevalley ordering on the rook monoid, *Turkish Journal of Math* **36** (2012), 499–519.
- [3] E. Egge, Enumerating rc-Invariant Permutations with No Long Decreasing Subsequences, *Annals of Combinatorics*, **14** (2010), pp. 85–101.
- [4] J. R. Stembridge, Some combinatorial aspects of reduced words in finite Coxeter groups, *Transactions of the American Mathematical Society*, **349**(4) (1997), 1285–1332.

This is joint work with Lara Pudwell.

Pattern Avoiding Involutions and the q -Analogues for Binomial Coefficients

Samantha Dahlberg

In their seminal paper, Simion and Schmidt showed that the cardinality of the set of involutions in the symmetric group $\mathfrak{S}_n$ which avoid the pattern 321 is a central binomial coefficient. We show that if one takes the generating function for the major index statistic over these involutions, then one obtains the usual q -analogue of this binomial coefficient. We will also discuss various properties of such polynomials for other patterns.

Patterns and Statistics in Set Partitions I

by

Samantha Dahlberg (Michigan State University)

Robert Dorward* (Oberlin College)

Jonathan Gerhard (James Madison University)

Thomas Grubb* (Michigan State University)

Carlin Purcell (Vassar College)

Lindsey Reppuhn* (Kalamazoo College)

Bruce E. Sagan (Michigan State University)

We consider partitions π of the set $[n] = \{1, 2, \dots, n\}$ and denote the set of all such partitions by Π_n . We use the notation $\pi = B_1/B_2/\dots/B_k \vdash [n]$ to indicate that $[n] = \uplus_i B_i$ (disjoint union) and call the B_i *blocks*. If $S \subseteq [n]$ and $\sigma \vdash [n]$, then the partition σ' of S obtained by intersecting the blocks of σ with S is called a *subpartition* of σ . We *standardize* σ' to a set partition $\pi \vdash [k]$, $k = |S|$, by replacing the smallest element of σ' with 1, the next smallest with 2, and so forth. In this case we say that σ *contains* π . We say that σ *avoids* π if σ does not contain π and let $\Pi_n(\pi) = \{\sigma \in \Pi_n : \sigma \text{ avoids } \pi\}$.

To obtain the *standard form* of $\sigma = B_1/B_2/\dots/B_k$ we list the blocks in the order so that $\min B_1 < \min B_2 < \dots < \min B_k$. Given σ in standard form, the *restricted growth function* (RGF) of σ is the word $w = w(\sigma) = w_1 w_2 \dots w_n$ where $w_i = j$ if $i \in B_j$. For example, if $\sigma = \{1, 4, 6\}/\{2, 3, 7\}/\{5\}$ then $w = w(\sigma) = 1221312$. Wachs and White introduced four fundamental statistics on set partitions via their RGFs and called them lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively. For example, $\text{lb}(w) = \sum_i \text{lb}_i(w)$ where $\text{lb}_i(w)$ is the number of integers to the left of w_i which are also bigger than w_i . (Note that multiple copies of the same integer are only counted once.) Continuing our example above, $\text{lb}(w) = 0 + 0 + 0 + 1 + 0 + 2 + 1 = 4$. We let $\text{lb}(\sigma) = \text{lb}(w(\sigma))$.

Letting q be a variable, we study the generating functions defined by $LB_n(\pi) = \sum_{\sigma \in \Pi_n(\pi)} q^{\text{lb}(\sigma)}$ for all $\pi \vdash [3]$ as well as the analogous polynomials for the other statistics. In particular, we determine their coefficients, degrees, and other properties.

Patterns and Statistics in Set Partitions II

by

Samantha Dahlberg (Michigan State University)

Robert Dorward (Oberlin College)

Jonathan Gerhard* (James Madison University)

Thomas Grubb (Michigan State University)

Carlin Purcell* (Vassar College)

Lindsey Reppuhn (Kalamazoo College)

Bruce E. Sagan (Michigan State University)

This talk is a continuation of Patterns and Statistics in Set Partitions I. Let P be a collection of set partitions. We consider the class $\Pi_n(P) = \{\sigma \in \Pi_n : \sigma \text{ avoids every } \pi \in P\}$. We will discuss the generating functions for the lb, ls, rb, and rs statistics over collections when $|P| > 1$ or when P contains a partition of more than three elements. Interesting connections arise with generalized Fibonacci numbers and noncrossing partitions.

Unsolved Problems In Computational Science

Shanzhen Gao

We will talk about some interesting open problems in computational science, including permutation patterns, partitions, lattice paths and walks. It seems that scientists in mathematics and computer science are still far away from solving them.

Probability Distributions Motivated by Permutations and Featuring Telescoping Series

Anant Godbole and Jie Hao, East Tennessee State University

In this poster, we study the simplest type of infinite telescoping sum, namely

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} (q_n - q_{n+1}) = 1,$$

due to our assumptions that $1 = q_1 \geq q_2 \geq q_3 \dots \geq 0$ and $\lim_{n \rightarrow \infty} q_n = 0$. Now the above clearly defines a probability distribution on $\mathbf{Z}^+$, but making unmotivated choices such as $q_n = \frac{1}{3n^4 - 2n^2}$ would serve little purpose, however, so let us state our objective: We wish to propose three probability distributions that each

- (i) are similar in form to familiar discrete distributions;
- (ii) arise from the cycle or pattern containment structure of random permutations; and
- (iii) lead to open-ended sets of questions.

Let us go one step further, by asking

- (iv) that there be events $A_n, n \geq 1$ in some probability space so that $q_n = P(A_n)$ with $A_{n+1} \subseteq A_n$ so that, as we shall see, $p_n, n \geq 1$ provides a model for *first occurrence distributions* of certain events.

The viewer of this poster should note that discrete models, other than those provided by random permutations, could equally well have been used as launching pads for our discrete distributions, and s/he is invited to come up with well-motivated first occurrence distributions along the lines of the ones in this paper.

Packing Set Partitions: A New Hope

Adam M. Goyt
Minnesota State University Moorhead
web.mnstate.edu/goytadam
goytadam@mnstate.edu

Abstract

A long time ago, at a Patterns Conference far, far away on the planet Hoth (aka Iceland), Jedi Master Stromquist took me under his wing to tutor me in the ways of pattern avoidance. After listening to my ramblings on pattern avoidance, and sensing the force of packing densities within me, he suggested that I undertake the study of packing set partitions¹. Over the next seven years I worked tirelessly as a Padawan learning the mystical ways of packing set partitions. In 2009, I was ready to face the trials of Knighthood, and sought the wisdom of the Council of Pattern Avoiders on Coruscant (Florence, Italy) to guide me on my quest. Unfortunately even the Council found the future of such a task clouded in uncertainty². Alas, I was left without even a Wookiee copilot to face this Numerical Menace of packing the simplest non layered set partition on my own. I spent the following years contemplating the teachings of my pattern master, honing my Jedi skills, and trying to force a solution. Despite all my efforts ‘to stay on target’, my solutions met with one failure after another until one evening in a Cantina in Glasgow, Princess Lara and I formulated an idea to destroy this Death Star of a problem once and for all. In this talk I present to you the final destruction of the case of set partition patterns of length three³.

¹The packing density $\delta(\pi)$ of a partition π of the set $[k] = \{1, 2, \dots, k\}$ is defined to be $\delta(\pi) := \lim_{n \rightarrow \infty} \frac{\mu(\pi, n)}{\binom{n}{k}}$, where $\mu(\pi, n)$ is the maximum number of copies of π in any partition of $[n]$.

²At this time I had determined the packing densities of the set partitions $1/2/3$, $1/23$, $12/3$, and 123 . They are $\delta(1/2/3) = 1$, $\delta(1/23) = \delta(12/3) = 2\sqrt{3} - 3$, and $\delta(123) = 1$.

³I will show that the packing density for $13/2$ is $\delta(13/2) = 1/4$.

Sorting index and Mahonian-Stirling pairs for labeled forests

Amy Grady

For a permutation $\sigma \in S_n$ there is a unique decomposition as a product of transpositions, $\sigma = (i_1, j_1)(i_2, j_2) \cdots (i_k, j_k)$, such that $j_1 < j_2 < \cdots < j_k$ and $i_1 < j_1, i_2 < j_2, \dots, i_k < j_k$. Recently, Petersen defined the sorting index of a permutation σ as follows

$$\text{sor}(\sigma) = \sum_{r=1}^k (j_r - i_r).$$

Petersen proved that the distribution of the sorting index is naturally related to three classical permutations statistics: the inversion index (inv), the number of right-to-left minima (rl-min), and the number of cycles (cyc). Namely,

$$\sum_{\sigma \in S_n} q^{\text{sor}(\sigma)} t^{\text{cyc}(\sigma)} = \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} t^{\text{rl-min}(\sigma)} = t(t+q) \cdots (t+q+q^2+\cdots+q^{n-1}).$$

The statistic inv was defined for labeled rooted forests by Björner and Wachs, who proved that the distribution of inv over all possible labelings of a forest P is

$$\sum_{w \in \mathcal{W}(P)} q^{\text{inv}(P,w)} = \frac{n!}{\prod_{x \in P} h_x} \prod_{x \in P} [h_x]$$

where h_x is the size of the subtree rooted at the vertex x . If P is a linear tree, the labelings of P when read from bottom to top correspond to permutations and the definition of $\text{inv}(P, w)$ reduces to the permutation inversion index.

We give definitions for bottom-to-top maxima, cycles, and sorting index for labeled forests that generalize the aforementioned permutation statistics. We then give explicit formulas for the generating functions of the pairs $(\text{inv}, \text{Bt-max})$ and (sor, Cyc) for a fixed forest and prove that they are equidistributed. Our proofs use two maps, A-code and B-code, between labeled forests and bounded integer sequences, which we construct.

We also define analogous statistics for signed labeled forests, and derive similar results about their distributions. Just like for the unsigned forests these results generalize the results for signed permutations.

THE REGULAR LANGUAGE OF SIMPLE PERMUTATIONS

RUTH HOFFMANN

A permutation pattern class is a downwards closed set of permutations, under the classical partial order of pattern involvement [4]. We will be specifically looking at classes of permutations that are regular languages under the rank encoding [2, 3, 5]. The *rank encoding* of a permutation $\pi = \pi(1)\pi(2)\dots\pi(n)$ is the sequence $E(\pi) = p_1p_2\dots p_n$ where for all $i \in \{1, \dots, n\}$,

$$p_i = \left| \left\{ x : x \in \{\pi(i), \pi(i+1), \dots, \pi(n)\}, x \leq \pi(i) \right\} \right| ,$$

is called the *rank* of $\pi(i)$, amongst the entries of π that have not occurred yet.

Following [2], the class of permutations with maximum rank $k \in \mathbb{N}$, will be denoted as Ω_k . In [2] Albert, Atkinson and Ruškuc, determined that $E(\Omega_k)$ is a regular language. A permutation pattern class $\mathcal{C}$ is a *regular pattern class* or a *regular class*, if $E(\mathcal{C})$ is a regular subset of $E(\Omega_k)$, for some finite $k \in \mathbb{N}$ being the size of the alphabet of the language.

An *interval* (or *block* [1]) in a permutation σ is a set of contiguous values of σ such that their indices are consecutive. It is easy to see that every permutation of length n has intervals of length 0, 1 and n , at least. The permutations of length n that *only* contain intervals of length 0, 1 and n are said to be *simple* [6].

Given a permutation π of length m and nonempty permutations $\alpha_1, \dots, \alpha_m$, the *inflation* of π by $\alpha_1, \dots, \alpha_m$, written as $\pi[\alpha_1, \dots, \alpha_m]$, is the permutation obtained by replacing each entry $\pi(i)$ by an interval that is order isomorphic to α_i , where the relative ordering of the intervals corresponds to the ordering of the entries of π [1]. Conversely a *block-decomposition* or *deflation* [1] of σ is any expression of σ as an inflation $\sigma = \pi[\alpha_1, \dots, \alpha_m]$.

Albert and Atkinson proved:

Proposition 1 ([1]). *Let σ be a permutation of finite length greater than 1. There is a unique simple finite permutation π , with $|\pi| > 1$ and a sequence $\alpha_1, \dots, \alpha_n$ of non-empty permutations such that*

$$\sigma = \pi[\alpha_1, \dots, \alpha_n].$$

If $\pi \neq 12, 21$ then $\alpha_1, \dots, \alpha_n$ are also uniquely determined by σ . If $\pi = 12$ or 21 , then α_1, α_2 are unique so long as we require that α_1 is plus-indecomposable or minus-indecomposable respectively.

We have successfully proven that the set of permutations of a regular class $\mathcal{C}$, that are plus- or minus-(in)decomposable, is also regular [7]. Furthermore, we have found that the set of permutations of a regular class $\mathcal{C}$, that are uniquely block-decomposed with the same simple permutation, is also regular. These were the first steps towards proving the following conjecture:

Conjecture 1. *Let $\mathcal{C}$ be a regular class with $\mathcal{C} \subseteq \Omega_k$, then the set $\mathcal{S} \subset \mathcal{C}$ containing all simple permutations of $\mathcal{C}$ is also regular.*

RUTH HOFFMANN

Currently we are in the process of proving this conjecture to be true, by utilising the tools from the block-decomposable languages mentioned above.

REFERENCES

- [1] M. H. Albert and M. D. Atkinson, *Simple permutations and pattern restricted permutations*, Discrete Mathematics, 300 (2005), pp. 1-15.
- [2] M. H. Albert, M. D. Atkinson and N. Ruškuc, *Regular closed sets of permutations*, Theoretical Computer Science, 306 (2003), pp. 85-100.
- [3] M. H. Albert, S. Linton and N. Ruškuc, *On the Permutational Power of Token Passing Networks*, tech. rep., 2004.
- [4] M. D. Atkinson, *Restricted Permutations*, Discrete Mathematics, 195 (1999), pp. 27-38.
- [5] M. D. Atkinson, M. J. Livesey and D. Tulley, *Permutations generated by token passing in graphs*, Theoretical Computer Science, 178 (1997), pp. 103-118.
- [6] R. Brignall, *A survey of simple permutations*, Permutation Pattern, 376 (2010), pp. 41-65.
- [7] R. Hoffmann and S. Linton, *Regular Languages of Plus- and Minus- (In)Decomposable Permutations*, Pure Mathematics and Applications (to Appear), (2013).

Enumeration of simple permutations in $\text{Av}(52341, 53241, 52431, 35142, 42513, 351624)$.

Masaki Ikeda

Abstract

ABSTRACT. Because the permutations of Schubert varieties defined by inclusions are contained in $\mathcal{A}' = \text{Av}(52341, 53241, 52431, 35142, 42513, 351624)$, we believe that the structures of simple permutations in this class are somewhat similar to the ones of simple permutations in $\text{Av}(4231, 35142, 42513, 351624)$. In this presentation, I describe the enumeration of simple permutations in $\mathcal{A}'$ whose extreme pattern is 2413 as well as their structural characterization.

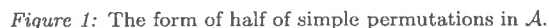
This work is still in progress for my doctoral dissertation. Proofs have not been carefully verified and in some parts still need some clarification.

Partial motivation for this work comes from algebraic geometry. In particular, certain classes of Schubert varieties are indexed by permutations in some pattern avoidance classes. For example, as shown by Lakshmibai and Sandhya [1], a smooth Schubert variety is indexed by a permutation in the class $\text{Av}(3412, 4231)$, and a Schubert variety defined by inclusions has an index in $\mathcal{A} = \text{Av}(4231, 35142, 42513, 351624)$, as shown by Gasharov and Reiner [2]. In addition to those results, Úlfarsson and Woo [3] showed Schubert varieties which are local complete intersections are indexed by permutations in $\mathcal{A}' = \text{Av}(52341, 53241, 52431, 35142, 42513, 351624)$. We call permutations in $\mathcal{A}'$ *almost defined by inclusions*, since this class is closely related to $\mathcal{A}$.

For each of these classes, it is natural to ask for its enumeration, that is, for each $n \geq 1$, how many permutations in S_n are contained in $\text{Av}(3412, 4231)$, $\mathcal{A}$ and $\mathcal{A}'$? The enumeration of the permutations indexing smooth Schubert varieties was initially found by Haiman [4], and then also discussed by Bousquet-Mélou and Butler [5]. Furthermore, Albert and Brignall [6] discovered the enumeration of Schubert varieties defined by inclusions. My presentation is about the enumeration of the class $\mathcal{A}'$.

The approach we take to enumerate $\mathcal{A}'$ is similar to the method used by Albert and Brignall to enumerate $\mathcal{A}$ in [6]. First, we investigate a structural characterization of the simple permutations in $\mathcal{A}'$, then we enumerate the whole class by inflation. Because $\mathcal{A}'$ is in some sense only slightly larger than $\mathcal{A}$, we believe that the structure of $\mathcal{A}'$ is similar to that of $\mathcal{A}$.

We now describe the approach of Albert and Brignall in [6]. The authors observe that essentially half of all simple permutations in $\mathcal{A}$ have the form shown in Figure 1. Then, they encode these permutations into a regular language and construct a finite state automaton accepting the language. This allows them to use the transfer matrix method to find the generating function for the number of simple permutations in $\mathcal{A}$. From this result and inflation with some adjustments, the generating function for general permutations in $\mathcal{A}$ can be found.



The extreme pattern of a permutation is the pattern defined by its first, last, greatest and least elements. In [6], it is shown that any simple permutation in $\mathcal{A}$ which has the extreme pattern 2413 is N -shaped as in Figure 2(a). In $\mathcal{A}'$, however, permutations can have extra segments as shown in Figure 2(b). For example, consider a simple permutation $\pi = 253794168$ in $\mathcal{A}'$. We can think of the segment between b and d as lying on two lines, one with the points for 3 and 7 and the other with the point for 5. As we can see, π is not in $\mathcal{A}$, since 5 and 3 are not in increasing order, and indeed, it contains 4231. The restriction to two (instead of more) parallel lines indicates that the first segment between b and d avoids 321. Indeed, this is required by the condition that the whole permutation avoids 53241.

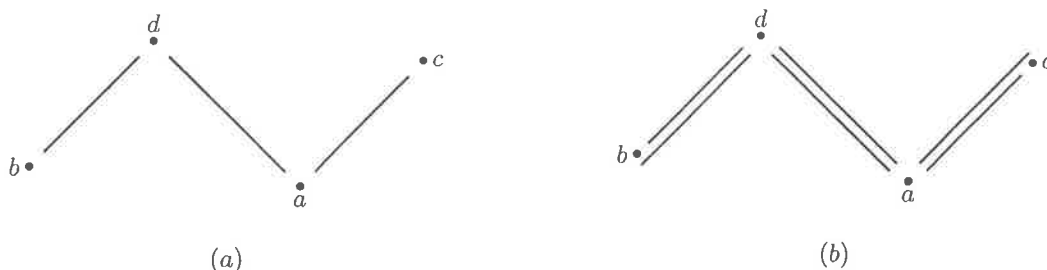


Figure 2: The form of simple permutations whose extreme patterns are 2413 in $\mathcal{A}$ and $\mathcal{A}'$ respectively.

Through encoding the precise details of this characterization by a regular language, constructing a finite state automaton, and using the transfer matrix method, we found (modulo some details to be carefully checked) that the generating function for the simple permutations in $\mathcal{A}'$ with extreme pattern 2413 is

$$f = -\frac{x^4(1 - x + 3x^2 + 6x^3 + 3x^4 + 3x^5 + 6x^6 + 5x^7 + 3x^8 + 3x^9 + x^{10})}{(1+x)(-1+3x+2x^4-2x^5-9x^6-6x^7+4x^9+6x^{10}+4x^{11}+x^{12})}.$$

The use of *PermLab* [7] has been extremely helpful to observe the structure of simple permutations in $\mathcal{A}'$. I would like to thank my advisor, Alexander Woo for the great support, and all the help towards my doctoral research as well as bringing my attention to Permutation Patterns 2014.

References

- [1] V. Lakshmibai, B. Sandhya, Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$, Proc. Indian Acad. Sci. Math. Sci. 100 (1990) 45-52.
- [2] V. Gasharov, V. Reiner, Cohomology of smooth Schubert varieties in partial flag manifolds, J. London Math. Soc. (2) 66 (2002) 550-562.
- [3] H. Úlfarsson and A. Woo, Which Schubert varieties are local complete intersections?, Proc. Lond. Math. Soc. (3) 107 (2013) 1004-1052.
- [4] M. Haiman, Smooth Schubert varieties, Preprint (1992).
- [5] M. Bousquet-Mélou, S. Butler, Forest-like permutations, Ann. Comb. 11 (2007) 335-354.
- [6] M. H. Albert, R. Brignall, Enumerating indices of Schubert varieties defined by inclusions, J. Combin. Theory Ser. A 123 (2013) 154-168.
- [7] M. H. Albert, PermLab: Software for Permutation Patterns (2012) <http://www.cs.otago.ac.nz/staffpriv/malbert/permlab.php>

Hoop Closure

Robert E. Jamison

We start with an ordered collection of n lists $L_1, \dots, L_n$, each of which contains the integers $1, \dots, n$. A subset H of one of these lists is a *hoop*. A choice of one hoop H_j for each list L_j is a *hoop system*. A permutation $\pi(1), \dots, \pi(n)$ *jumps through* the hoops H_j of a hoop system H iff $\pi(j) \in H_j$ for all j . Such a permutation can be regarded as a system of distinct representatives (SDR) for the family of hoops. For each hoop system H , let $\Pi(H)$ denote the set of all permutations that jump through the hoops of H . These sets form the closed sets of a closure system $\mathcal{H}$ on the permutations of $1, \dots, n$. This talk will discuss properties of this closure system.

Apologia I have chosen this topic because it caught my attention several years before I retired and because it deals with closure systems, an interest dating back to my dissertation. The following background material comes from my book on *An Introduction to Closure Systems*.

Closure Systems A *closure space* $(X, \mathcal{C})$ consists of a *ground set* X and a family $\mathcal{C}$ of *closed* subsets of X . It is required that the arbitrary intersection of closed sets be closed. Since we are working in a finite situation, it suffices to say that the intersection of any two closed sets is closed. The intersection of all closed sets containing a subset S of X is the *closure* of S and is denoted by $\mathcal{C}(S)$. All closure operators φ satisfy these three properties:

- 1) (*isotone*) $A \subseteq B \implies \varphi(A) \subseteq \varphi(B)$;
- 2) (*expansive*) $S \subseteq \varphi(S)$ for all subsets S of X ;
- 3) (*idempotent*) $\varphi(\varphi(S)) = \varphi(S)$ for all subsets S of X .

The closed sets in a topology and the convex sets in Euclidean space form familiar examples of closure systems. There are many algebraic examples as well as such examples as the subgroups of a group, subrings of a ring, ideals in a ring, and so forth. These algebraic examples have the added property that the union of a chain of closed sets is again closed. Systems with this chain union property are *alignments*. Again the chain union property holds trivially in finite spaces, so the system described above is the **hoop alignment** $\mathcal{H}$ of order n .

In addition to the topological and algebraic examples just mentioned, there are also combinatorial closure systems, matroids and antimatroids being the best known. The hoop alignment $\mathcal{H}$ is interesting because it is a natural combinatorial alignment, but it is neither a matroid nor an antimatroid. Our goal will be to study several closure theoretic invariants for hoop closure. Here are two such properties.

If a set contains the closure of any two of its points, is it closed? This is a familiar property from Euclidean geometry: if a set contains the line segment between any two of its points, then it is convex. That is, closure (aka convexity) is determined by a two-point condition. This does not always hold. For example, the graphic matroid of a graph has this property iff the graph is chordal. I.e., there are no induced cycles of length 4 or more.

The second property is the analog of a classical theorem of Carathéodory:

In $\mathbb{R}^d$ if $p \in \text{conv}(S)$, then there is a subset F of S with at most $d+1$ points such that $p \in \text{conv}(F)$.

The Carathéodory number of a closure space $(X, \mathcal{C})$ is the smallest integer c such that

for any set S and point p , if $p \in \mathcal{C}(S)$, then there is a subset F of S with at most c points such that $p \in \mathcal{C}(F)$.

Thus Carathéodory's theorem says the Carathéodory number of the standard convexity on $\mathbb{R}^d$ is $c = d+1$.

The operator Π defined above carries hoop systems into sets of permutations. There is a reverse map Ψ that carries sets of permutations into hoop systems. Namely, if S is a set of permutations, then $\Psi(S)$

Table 1: Schematic for S

$\beta\gamma\delta$	α				
	$\gamma\delta$	$\alpha\beta$			
		δ	γ	β	α
		γ	$\alpha\beta\delta$		
	β			$\alpha\gamma\delta$	
α					$\beta\gamma\delta$

Table 2: Template of S

Hoops	H_1	H_2	H_3	H_4	H_5	H_6
indices/	$\{1, 6\}$	$\{1, 2, 5\}$	$\{2, 3, 4\}$	$\{3, 4\}$	$\{3, 5\}$	$\{3, 6\}$
1	*	*				
2		*	*			
3			*	*	*	*
4			*	*		
5		*			*	
6	*					*

has hoops H_j defined by $H_j := \{\pi(j) : \pi \in S\}$. The operator Π is isotone: the larger the hoops, the more permutations can jump through them. The operator Ψ is also isotone: the more permutations, the larger the hoops that they define. Starting with a set S of permutations, the composition $S \longrightarrow \Psi(S) \longrightarrow \Pi(\Psi[S])$ produces the hoop closure of the set S . In particular, it is a isotone and expansive. Such set maps were called *hulls* by Hausdorff. Going in the opposite direction starting with a hoop system H , the composition $H \longrightarrow \Pi(H) \longrightarrow \Psi(\Pi[H])$ is again isotone, but the hoops may shrink in the process. Such set maps were called *kernels* by Hausdorff.

Examples Let us now look at some explicit examples. Let S be the set of these four permutations of 123456:

$$\alpha : 612453 \quad \beta : 152436 \quad \gamma : 124356 \quad \delta : 123456$$

The first table shows the *schematic* of these four permutations. The names of the permutations are shown in the cells of a matrix subject to the rule that a permutation symbol π appears in row (i, j) iff the j th term of π is i — that is, $\pi(j) = i$.

The template is derived from the schematic by suppressing the permutation names but recording the positions where they occur. In this case I have added the hoops of the hoop system $\Psi(S)$ across the top.

The next three tables show what happens if we try to build a permutation that jumps through the given hoops. Let us consider the cell 61. The permutation α uses this cell, so it is possible to build a permutation that starts with 61. We want to show that α is the *only* such permutation. If we choose cell 61, then cell 66 cannot be used. Thus to cover the last column, 36 is the only choice. Now the choice of 36 excludes using 33, 34, 35. Continuing in this way, we see that we must end up with α again.

It follows that if α is removed from the set, then square 61 will disappear from the template. That is, if we take α out, we can't get it back. Such a point is an *extremepoint*. Formally, if C is a closed set in a closure space $(X, \mathcal{C})$, then $p \in C$ is an *extremepoint* of C iff $C \setminus p$ is again closed. The vertices of convex polygons are their extremepoints. This notion of extremepoint is crucial to the theory of antimatroids.

This final schematic shows a hoop system where there are no extremepoints. Hence hoop closure is not an antimatroid.

Table 3: Building a Permutation Starting with $\alpha(1)$

*	$\alpha(2)$				
	$\gamma\delta$	$\alpha\beta$			
		δ	γ	β	$\alpha(6)$
		γ	$\alpha\beta\delta$		
	β			$\alpha\gamma$	
$\alpha(1)$					*

Table 4: Building a Permutation - 2nd step

*	$\alpha(2)$				
	*	$\alpha(3)$			
		*	*	*	$\alpha(6)$
		γ	$\alpha\beta\delta$		
	*			$\alpha(5)$	
$\alpha(1)$					*

Table 5: Building a Permutation - 3rd step

*	$\alpha(2)$				
	*	$\alpha(3)$			
		*	*	*	$\alpha(6)$
		*	$\alpha(4)$		
	*			$\alpha(5)$	
$\alpha(1)$					*

Table 6: No Extremepoints

α	β	γ		
β	γ	α		
γ	α	β		
			$\alpha\beta$	γ
			γ	$\alpha\beta$

Hook Coefficients of Chromatic Symmetric Functions

Ryan Kaliszewski
Department of Mathematics
Drexel University

The chromatic symmetric function associated to a graph G is introduced in [Stanley(1995)] as a generalization of the chromatic polynomial. When the chromatic symmetric function is expanded in various bases of symmetric functions the coefficients are related to partitions of the vertices into stable subsets, the Möbius function of the lattice of contractions of G , and the structure of the acyclic orientations of G [Greene and Zaslavsky(1983), Stanley(1995), Whitney(1932)]. Also, for certain graphs these coefficients are related to Hecke algebras and Kazhdan Lusztig polynomials [Haiman(1993), Stanley(1995)].

Since the chromatic symmetric function is indeed symmetric, it can be expanded in the basis of Schur functions, $X_G = \sum_{\lambda} c_{\lambda}^s s_{\lambda}$. If we denote the hook $\lambda^{(k)} = (k, 1, \dots, 1)$ we can refer to $c_{\lambda^{(k)}}^s$ as a *hook coefficient*. This paper will show that the hook coefficients are (positive) weighted enumerations of the acyclic orientations of G . Specifically, we will prove the following theorem.

Theorem 0.1. *If a_j denotes the number of acyclic orientations of the graph G with j sinks, then*

$$c_{\lambda^{(k)}}^s = \sum_{j=1}^n \binom{j-1}{k-1} \cdot a_j \quad (1)$$

where n is the number of vertices of G .

Previously it was proven by Gasharov [Gasharov(1996)] that if G is the incomparability graph of a poset and G is claw-free, then all of the coefficients in the Schur expansion, c_{λ}^s , are non-negative. It was conjectured by Gasharov that for any graph that is claw-free all coefficients in the Schur expansion are non-negative. Further, though in [Chow(1997)] formula (1)

was shown for incomparability graphs of (finite) posets, this formula was independently rediscovered for *any* graph by using Stanley's quasisymmetric construction.

Also, in the case of the claw-free incomparability graph it has been proven by Gasharov [Gasharov(1996)] that the coefficient c_λ^s is equal to the number of P -tableaux of shape λ . There is a correspondence between P -tableaux of hook shapes and acyclic orientations of the incomparability graph presented in [Chow(1997)]. In this paper that correspondence will be presented in a language closer to the proof of formula (1) and the inverse of the correspondence will be presented.

1 Background

Given a permutation $\sigma \in \mathfrak{S}_n$ define the *descent set*, $Des(\sigma)$, to be $i \in \{1, 2, \dots, n-1\}$ such that $\sigma^{-1}(i+1) < \sigma^{-1}(i)$, or $i+1$ occurs before i in the word of σ . If $Des(\sigma) = \{i_1 < i_2 < \dots < i_k\}$ define the composition $Des'(T) = (i_1, i_2 - i_1, i_3 - i_2, \dots, n - i_k) \models n$.

The ring of symmetric functions has several well-known bases related to tableaux, see [Stanley(1999)] or [Fulton(1997)]. We will now describe a relationship between symmetric functions and quasisymmetric functions.

Definition Given a permutation $\sigma \in \mathfrak{S}_n$ define the **fundamental quasisymmetric function** on an alphabet x to be

$$F_{Des(\sigma)}(x) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ i_j = i_{j+1} \Rightarrow i_j \notin Des(\sigma)}} x_{i_1} \cdots x_{i_n}. \quad (2)$$

Note that by the correspondence outlined above, we can index fundamental quasisymmetric functions by compositions.

If $\{b_\alpha\}_{\alpha \in I}$ is a basis of the (quasi-)symmetric functions over an indexing set I , we can expand a function in terms of the b_λ . We will write such an expansion

$$f = \sum_{\alpha} c_{\alpha}^b \cdot b_{\alpha}(x). \quad (3)$$

If each $c_{\alpha}^b \geq 0$ then we say that f is b -positive.

In general we will reserve lowercase letters for symmetric functions and capital letters for quasisymmetric functions.

2 Chromatic Symmetric Functions

Throughout this section let $G = (V, E)$ be a graph with finite vertex set V and finite edge set E . We will also assume that G is simply laced and has no loops.

Definition A **coloring** of a graph, G , is a map $\kappa : V \rightarrow \mathbb{N}$. The numbers, $\kappa(V)$ are referred to as the ‘colors.’

Definition A **proper coloring** of a graph, G , is a coloring such that for all $uv \in E$, $\kappa(u) \neq \kappa(v)$. That is, adjacent vertices never share the same color.

Definition Let κ be a proper coloring. The **coloring function** (or coloring monomial) associated to κ is

$$x^\kappa = \prod_{v \in V} x_{\kappa(v)}. \quad (4)$$

The **chromatic symmetric function** of G is defined as

$$\mathbf{X}_G = \mathbf{X}_G(x) = \sum_{\kappa} x^\kappa \quad (5)$$

where the sum is over all proper colorings, κ .

Observe that for any coloring, κ , and any permutation, $\sigma \in \mathfrak{S}_n$, $\sigma \circ \kappa$ is also a coloring of G . Thus this shows that $\mathbf{X}_G$ is indeed symmetric. Furthermore, it is immediate that $\mathbf{X}_G$ is homogeneous of degree n .

3 Hook Coefficients of the Chromatic Symmetric Function

Definition A **hook** is a partition of the following form: $(k, 1, 1, \dots, 1)$, where $k \in \mathbb{N}$. When we expand $\mathbf{X}_G$ in the basis of Schur functions a **hook coefficient** is any coefficient of a Schur function with a hook parameter.

Example 3.1. The chromatic symmetric function of the claw graph is

$$X_{K_{3,1}} = s_{(3,1)} - s_{(2,2)} + 5s_{(2,1,1)} + 8s_{(1,1,1,1)}. \quad (6)$$

The hook coefficients are 1 for the hook $(3, 1)$, 5 for the hook $(2, 1, 1)$, and 8 for the (trivial) hook $(1, 1, 1, 1)$.

Let $\lambda^{(k)}$ be the hook $(k, 1, \dots, 1)$ with length $n - k + 1 \geq 1$ and let $a_k(G) = a_k$ be the number of acyclic orientations of G with k sinks.

Theorem 3.2. *The hook coefficient for $\lambda^{(k)}$ is*

$$c_{\lambda^{(k)}}^s = \sum_{j=1}^n \binom{j-1}{k-1} \cdot a_j. \quad (7)$$

Proof of Theorem 3.2

We will break the proof of Theorem 3.2 into several sections. Throughout the proof, fix a $k \in \mathbb{N}$, let $\lambda^{(k)} \vdash n$, and set $N = n - k + 1 = \ell(\lambda^{(k)})$.

3.0.1 Writing X_G in the Quasisymmetric Basis

In this section we will present Stanley's construction of X_G in the basis of fundamental quasisymmetric functions, F_α [Stanley(1995)].

Lemma 3.3. [Egge and Warrington(2010)] *A Schur hook coefficient is equal to the corresponding quasisymmetric hook coefficient. That is,*

$$c_{\lambda^{(k)}}^s = c_{\lambda^{(k)}}^F. \quad (8)$$

Let σ be an acyclic orientation of the graph G , i.e. an assignment of an arrow to each edge such that there are no cycles in the resulting directed graph. For each acyclic orientation, σ , fix an order-reversing map ω_σ . Let A_σ be the set of all order-preserving maps of σ . By [Stanley(1995)]

$$X_G = \sum_{\sigma} \sum_{\alpha \in A_\sigma} F_{Des'(\alpha \omega_\sigma^{-1})}. \quad (9)$$

Remark Note that $\alpha \omega_\sigma^{-1}$ is necessarily a permutation.

3.0.2 Quasisymmetric Hook Coefficients

Up until now we have been considering the Schur hook coefficients, $c_{\lambda^{(k)}}^s$. Let us turn our attention to the quasisymmetric hook coefficients, or more specifically $c_{\lambda^{(k)}}^F$.

Construct the set

$$\Omega_{n,k} = \{\sigma \in \mathfrak{S}_n \mid F_{Des'(\sigma)} = F_{\lambda^{(k)}}\} \quad (10)$$

$$= \{\sigma \in \mathfrak{S}_n \mid Des(\sigma) = \{k, k+1, \dots, n\}\} \quad (11)$$

Note that from lines (9) and (10) we can deduce that

$$c_{\lambda(k)}^F = \#\{\text{permutations of the form } \alpha\omega_o^{-1} \text{ that lie in } \Omega_{n,k}\} \quad (12)$$

$$= \#\{A_o\omega_o^{-1} \cap \Omega_{n,k}\}. \quad (13)$$

Now consider that for a permutation, ρ , to have descent set $\{k, k+1, \dots, n\}$ it must be true that

$$\rho^{-1}(1) < \rho^{-1}(2) < \dots < \rho^{-1}(k), \quad (14)$$

and

$$\rho^{-1}(n) < \rho^{-1}(n-1) < \dots < \rho^{-1}(k). \quad (15)$$

Thus if we write ρ in inline notation, it *must* end with k . The entries $1, 2, \dots, k-1$ can be placed anywhere so long as they are increasing from left to right. The remaining entries $n, n-1, \dots, k+1$ must be placed in the leftover slots so that they are decreasing.

3.0.3 The Set $A_o\omega_o^{-1} \cap \Omega_{n,k}$

Now we construct exactly the set of permutations that arise from $A_o\omega_o^{-1}$ that have descent set $\{k, k+1, \dots, n\}$. Fix an acyclic orientation o and let j be the number of sinks of o . In Stanley's construction we can choose ω_o to be any order-reversing map, so let us choose a map such that $\omega_o^{-1}(i)$ is a sink for every $i > n-j$. Furthermore, this choice of ω_o gives a total ordering on the vertices,

$$u >_\omega v \iff \omega_o(u) > \omega_o(v). \quad (16)$$

Let $s = \omega_o^{-1}(n)$. This "special sink" will be used in construction of subcollections of sinks to help enumerate those $\alpha \in A_o$ such that $\alpha\omega_o^{-1} \in \Omega_{n,k}$.

Enumerate a set of $k-1$ sinks, respecting the order given by ω_o with $v_1, \dots, v_{k-1}$ and enumerate the remaining vertices with $u_1, \dots, u_{n-k}$:

$$u_1 <_\omega u_2 <_\omega \dots <_\omega u_{n-k}, \quad v_1 <_\omega v_2 <_\omega \dots <_\omega v_k = s. \quad (17)$$

Define a map $\beta_v : V \rightarrow \{1, \dots, n\}$ as follows:

$$\beta_v(x) = \begin{cases} i & \text{if } x = v_i \\ n-i+1 & \text{if } x = u_i \end{cases} \quad (18)$$

Proposition 3.4. *The function β_v is order-preserving for any such enumeration of $k-1$ sinks.*

Proposition 3.5. *The descent set of the permutation $\beta_v \omega_0^{-1}$ is $\{k, k + 1, \dots, n\}$ for all such enumerations of $k - 1$ sinks.*

Proposition 3.6. *If $\alpha \in A_0$ is an order-preserving map such that $\text{Des}(\alpha \omega_0^{-1}) = \{k, k + 1, \dots, n\}$ then there is an enumeration of sinks such that $\alpha = \beta_v$.*

Proposition 3.7. *There are $\binom{j-1}{k-1}$ order preserving maps, β_v .*

For complete proofs see [Kaliszewski(2014)].

References

- [Bender and Knuth(1972)] E. A. Bender and D. E. Knuth. Enumeration of plane partitions. *J. Combin. Theory Ser. A*, 13:40–54, 1972.
- [Chow(1997)] T. Y. Chow. A note on a combinatorial interpretation of the e-coefficients of the chromatic symmetric function. *ArXiv Mathematics e-prints*, Dec. 1997.
- [Egge and Warrington(2010)] N. L. Egge, E. and G. Warrington. From quasisymmetric expansions to schur expansions via a modified inverse kostka matrix. *European J. Combin.*, 31:2014–2027, 2010.
- [Fulton(1997)] W. Fulton. *London Mathematical Society Student Texts*, volume 35, chapter Young tableaux, with applications to representation theory and geometry. Cambridge University Press, Cambridge, 1997.
- [Gasharov(1996)] V. Gasharov. Incomparability graphs of $(3+1)$ -free posets are s-positive. *Discrete Math*, 157, 1996.
- [Gessel(1984)] I. M. Gessel. Multipartite P-partitions and inner products of skew schur functions. *Contemp. Math*, 34:289–317, 1984.
- [Greene and Zaslavsky(1983)] C. Greene and T. Zaslavsky. On the interpretation of whitney numbers through arrangements of hyperplanes, zonotopes, non-radon partitions, and orientations of graphs. *Trans. Amer. Math. Soc.*, 280:97–126, 1983.
- [Haiman(1993)] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, 6:596–595, 1993.
- [Kaliszewski(2014)] R. Kaliszewski. Hook coefficients of chromatic symmetric functions. *ArXiv e-prints*, Apr. 2014.

- [Knuth(1970)] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math*, 34:709–727, 1970.
- [Loehr and Warrington(2012)] N. A. Loehr and G. S. Warrington. Quasisymmetric expansions of schur function plethysms. *Proc. Amer. Math. Soc.*, 140(4):1159–1171, 2012.
- [Stanley(1995)] R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, III:166–194, 1995.
- [Stanley(1998)] R. P. Stanley. Graph colorings and related symmetric functions: ideas and applications. *Discrete Mathematics*, 193:267–286, 1998.
- [Stanley(1999)] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, 1999. Volume 62 of Cambridge Studies in Advanced Mathematics.
- [Witney(1932)] H. Witney. A logical expansion in mathematics. *Bull. Amer. Math. Soc.*, 38, 1932.

PATTERN OCCURRENCES IN RANDOM WALKS

MEGAN MARTINEZ
DARTMOUTH COLLEGE

1. PATTERNS AND RANDOM WALKS

In this talk, we explore patterns in random walks in discrete time. To construct a random walk of length n , we pull $n - 1$ independent and identically distributed (IID) random variables, $X_1, X_2, X_3, \dots, X_{n-1}$ from some distribution. We let X_i be the step size of the i^{th} step in our random walk and let Z_i denote the value of the walk after i steps (we set $Z_0 = 0$). So, we let $Z_i = \sum_{j=1}^i X_j$.

Our goal is to investigate the notion of a pattern in a random walk. A length n random walk, $Z_0, Z_1, Z_2, \dots, Z_{n-1}$ corresponds to a pattern $\pi \in S_n$ where $\pi(i) = |\{k \mid Z_k < Z_j\}| + 1$. If $Z_i = Z_j$ for some $i, j \in [0, n - 1]$ where $i \neq j$, then there is no corresponding pattern. Since the random walk is completely dependent on the random variables $X_1, X_2, \dots, X_{n-1}$, we denote the correspondence omitting the Z_i 's. Define this mapping to patterns as $p: \mathbf{R}^{n-1} \rightarrow S_n$ where $p(X_1, X_2, \dots, X_{n-1}) = \pi$ and p is undefined if there exists some $i, j \in [0, n - 1]$ with $i \neq j$ such that $\sum_{k=1}^i X_k = \sum_{k=1}^j X_k$. For example $p(13, -8.7, -6.4, -0.9, 17) = 354216 \in S_6$ since 354216 is the relative order of the sequence of partial sums 0, 13, 4.3, -2.1, -3, 14.

Random walks constructed in this manner occur in a number of fields. Most notably, they are used in economics where the "random walk hypothesis" states that stock prices follow a random walk and therefore cannot be predicted. In recent years, the concept of forbidden patterns in the behavior of stock prices has been used in financial analysis [1]. Our research is an investigation from a combinatorial perspective into the behavior of pattern occurrences in a perfectly random walk.

We pose a global question: when do two patterns have equal probability of occurring regardless of probability distribution? Precisely, for some $\pi \in S_n$, we are interested in the set of permutations $\{\tau \in S_n \mid \text{Prob}(p(X_1, X_2, \dots, X_{n-1}) = \tau) = \text{Prob}(p(Y_1, Y_2, \dots, Y_{n-1}) = \pi) \text{ for every probability distribution}\}$ where $X_1, X_2, \dots, X_{n-1}$ and $Y_1, Y_2, \dots, Y_{n-1}$ are sequences of IID random variables. We want to be able to partition S_n into classes such as these.

In this talk we will propose a finer partition than the one above, and completely characterize the classes in this new partition.

To define the classes, we first describe every permutation using a canonical set of inequalities on the random variables $X_1, X_2, \dots, X_{n-1}$. The inequalities are defined in such a way that $p(X_1, X_2, \dots, X_{n-1}) = \pi$ if and only if $X_1, X_2, \dots, X_{n-1}$ satisfy the canonical inequalities associated to π .

For any permutation π , define a canonical set of $n - 1$ inequalities associated to π where the i^{th} inequality is defined as follows:

- if $\pi^{-1}(i) > \pi^{-1}(i + 1)$ then the i^{th} inequality is $\sum_{j=\pi^{-1}(i+1)}^{\pi^{-1}(i)-1} X_j < 0$.
- if $\pi^{-1}(i + 1) > \pi^{-1}(i)$ then the i^{th} inequality is $\sum_{j=\pi^{-1}(i)}^{\pi^{-1}(i+1)-1} X_j > 0$.

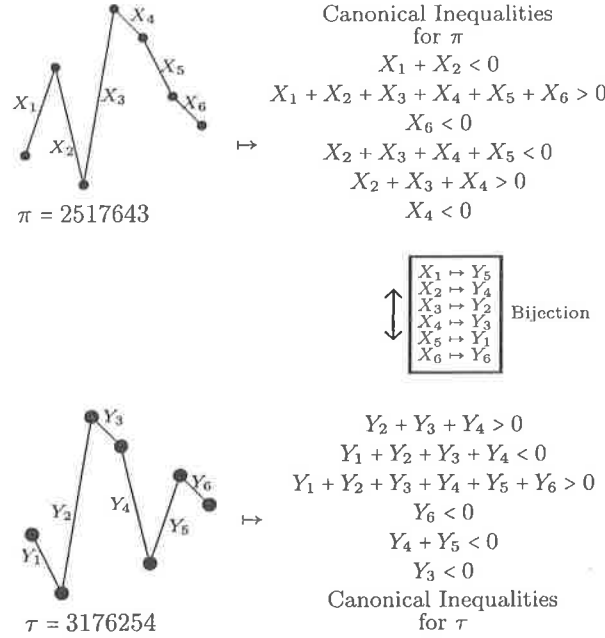


FIGURE 1. The canonical inequalities for $\pi = 2517643$ and $\tau = 3176254$ are equivalent up to bijection.

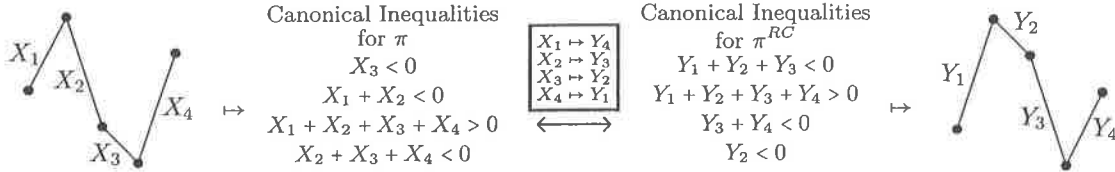


FIGURE 2. $\pi = 35214$ and its reverse-complement $\pi^{RC} = 25413$

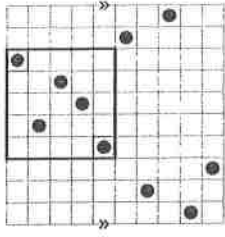
We say that two permutations have the same canonical inequalities up to bijection if we can map the inequalities of one to another with a bijection on the variables (the order of the inequalities doesn't matter). For an example see Figure 1.

If two permutations have the same canonical inequalities up to bijection then those permutations have the same probability of occurring regardless of probability distribution. We therefore define classes $\mathcal{C}(\pi) = \{\tau \in S_n \mid \text{the canonical inequalities of } \pi \text{ and } \tau \text{ are equivalent up to bijection}\}$. Our result completely characterizes the classes $\mathcal{C}(\pi)$.

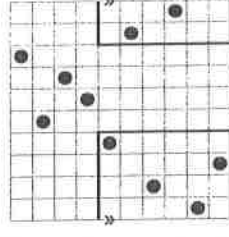
First notice that $\pi^{RC} \in \mathcal{C}(\pi)$: the canonical inequalities for π are the same up to bijection as the canonical inequalities for π^{RC} (we use Y_i 's instead of X_i 's for the inequalities of π^{RC}). We can see this by applying the bijection where $X_1 \mapsto Y_{n-1}$, $X_2 \mapsto Y_{n-2}$, $X_3 \mapsto Y_{n-3}, \dots, X_{n-1} \mapsto Y_1$ (Figure 2)

Applying a bijection to a set of inequalities is ultimately a transformation on the associated permutation. The bijection between the canonical inequalities of π and π^{RC} is a fairly simple construction: the first variable maps to the last, the second maps to the second to last, and so on. This mapping translates to a 180° flip of π to obtain π^{RC} .

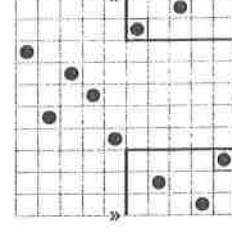
It turns out that all bijections between canonical inequalities can be expressed as a composition of bijections similar to that of the reverse-complement. We are able to describe the transformation on the associated permutation and use this to characterize the classes $\mathcal{C}(\pi)$.



(A) The bordered cylindrical block 85764



(B) The unbordered cylindrical block 4, 9, 2, 10, 1, 3



(C) The bordered cylindrical block 9, 2, 10, 1, 3

FIGURE 3. Examples of bordered and unbordered cylindrical blocks in $\pi = 8, 5, 7, 6, 4, 9, 2, 10, 1, 3$

2. A TRANSFORMATION ON π

The transformation is an action directly on the permutation $\pi = \pi(1)\pi(2)\pi(3)\dots\pi(n)$ and will produce another permutation in $\mathcal{C}(\pi)$.

Recall that a grid diagram for a permutation π is a grid where we label the bottom left box as $(0, 0)$, fill the $(i, \pi(i))$ cell for all $i \in [n]$, and leave all other cells empty. A block in a permutation π is a consecutive substring $\pi(i)\pi(i+1)\pi(i+2)\dots\pi(i+k)$ such that $\{\pi(i), \pi(i+1), \pi(i+2), \dots, \pi(i+k)\} = \{j, j+1, j+2, \dots, j+k\}$, which we abbreviate as $\pi([i, i+k]) = [j, j+k]$, for some $j \in [n-k]$. For example, when $\pi = 2647513$, $\pi([2, 5]) = [4, 7]$. So 6475 is a block in π . We allow singleton blocks and the block $\pi(1)\pi(2)\pi(3)\dots\pi(n)$, which will always exist. Notice that when we represent a block on a grid, the regions left, right, above, and below the block are empty.

In order to explain the transformation on π , we need to define a generalization of a block, which we call a "cylindrical block." We consider the grid as embedded onto the surface of a cylinder and identify the top and bottom edges of the diagram. A cylindrical block is analogous to a block, except the block is allowed to split across the top and bottom of the grid (though does not have to – hence blocks are also cylindrical blocks).

Definition 2.1. In a permutation $\pi \in S_n$, a **cylindrical block** is a consecutive substring $\pi(i)\pi(i+1)\pi(i+2)\dots\pi(i+k)$ such that one of the following is true:

- (1) there exists some $j \in [n-k]$ such that $\pi([i, i+k]) = [j, j+k]$
- (2) there exists some $j \in [k]$ such that $\pi([i, i+k]) = [1, j] \cup [n-k+j, n]$

Notice that the first case gives us the definition of a traditional block. The second case covers the situations where a block spans the "merged" edge. As an example consider the permutation $\pi = 8, 5, 7, 6, 4, 9, 2, 10, 1, 3$. There are 14 nontrivial cylindrical blocks in this permutation, one of which is 4, 9, 2, 10, 1, 3 (Figure 3b).

Definition 2.2. A cylindrical block $\pi(i)\pi(i+1)\pi(i+2)\dots\pi(i+k)$ is said to be **bordered** if

- (1) when $\pi([i, i+k]) = [j, j+k]$, $\{\pi(i), \pi(i+k)\} = \{j, j+k\}$
- (2) when $\pi([i, i+k]) = [1, j] \cup [n-k+j, n]$, $\{\pi(i), \pi(i+k)\} = \{j, n-k+j\}$

Our transformation acts on bordered cylindrical blocks.

Definition 2.3. Given a permutation $\pi \in S_n$, we say that another permutation $\pi' = \pi'(1)\pi'(2)\dots\pi'(n)$ is obtained by **flipping** the bordered cylindrical block $\pi(i)\pi(i+1)\pi(i+2)\dots\pi(i+k)$ if $\pi'(t)$ is defined as follows:

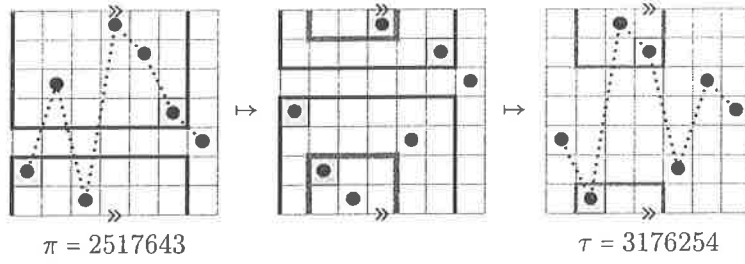


FIGURE 4. The sequence of flips of bordered cylindrical blocks that maps $\pi = 2517643$ to $\tau = 3176254$

- (1) when $\pi([i, i+k]) = [j, j+k]$,

$$\pi'(t) = \begin{cases} 2j+k-\pi(2i+k-t), & \text{if } t \in [i, i+k] \\ \pi(t), & \text{if } t \in [1, i-1] \cup [i+k+1, n] \end{cases}$$

- (2) when $\pi([i, i+k]) = [1, j] \cup [n-k+j, n]$,

$$\pi'(t) = \begin{cases} n-\pi(2i+k-t)+1, & \text{if } t \in [i, i+k] \\ \pi(t)+k-2j+1, & \text{if } t \in [1, i-1] \cup [i+k+1, n] \end{cases}$$

Graphically, a flip of a bordered cylindrical block is akin to rotating the block by 180° , which is similar to the effect of taking the reverse-complement.

The canonical inequalities of π are in bijection with the canonical inequalities of π' by the mapping where $X_i \mapsto Y_{i+k-1}$, $X_{i+1} \mapsto Y_{i+k-2}$, $X_{i+2} \mapsto Y_{i+k-3}, \dots$, $X_{i+k-1} \mapsto Y_i$, and $X_t \mapsto Y_t$ for all $t \in [1, i-1] \cup [i+k, n-1]$. Notice that this mapping is very similar to the one used with the reverse-complement.

Theorem 2.4. *Let $\pi, \tau \in S_n$. Then $\tau \in \mathcal{C}(\pi)$ if and only if τ can be obtained from π by a sequence of flips of bordered cylindrical blocks and the reverse-complement operation.*

Figure 4 gives an example of our theorem in action, showing a sequence of flips for two inequalities we know are in the same class (by Figure 1).

The proof of Theorem 2.4 involves considering all possible bijections from a set of canonical inequalities. We then work to determine which bijections yield another canonical set of inequalities and characterize those bijections. In order to do this, we associate each set of inequalities to a pictorial representation akin to a graph. Our argument finds a nested structure in the diagrams and uses an inductive argument that exploits this nesting.

This is joint work with Sergi Elizalde. We thank Adeline Pierrot for contributions in the early stages of this work.

REFERENCES

- [1] M. Zanin, *Forbidden Patterns in Financial Time Series*. Chaos 18 (2008), 013119.

Medians of permutations: when $A = \mathcal{M}(A)$ *

Sylvie Hamel and Robin Milosz
Département d'Informatique et de Recherche Opérationnelle
Université de Montréal - Québec, Canada
{sylvie.hamel,robin.milosz}@umontreal.ca

Abstract

Let $A = \{\pi_1, \pi_2, \dots, \pi_m\}$ be a set of m permutations of $\{1, 2, \dots, n\}$. The set of medians of A , denoted $\mathcal{M}(A)$, is the set of all permutations in $\mathcal{S}_n$ that are the "closest" of A under a distance function d . Here we investigated the case $A = \mathcal{M}(A)$ under the Kendall- τ distance.

1 Introduction

The problem of finding medians of a set of m permutations of $\{1, 2, \dots, n\}$ under the Kendall- τ distance [8, 12] is often cited in the literature as the Kemeny Score Problem [7]. In this problem m voters have to order a list of n candidates according to their preferences. The problem then consist in finding a *Kemeny consensus*; an order of the candidates that agrees the most with the order of the m voters, *i.e.*, that minimizes the sum of the disagreements.

This problem is polynomial-time solvable for $m = 2$, has been proved to be NP-complete when $m \geq 4$ [6] but its complexity remains unknown for $m = 3$. In the last 10 years, different approximation algorithms have been derived [1, 2, 3, 5, 9, 13]. In [4] we began to study the problem in a more combinatorial point a view, deriving some nice properties of the medians of a set of m permutations, with m odd.

In this extended abstract, we give a first investigation of the case $A = \mathcal{M}(A)$ *i.e.* we want to know exactly when a set of permutations is equal to the set of its medians. The abstract is organized as follow. After introducing the basic definitions and notations (in Section 2), we provide, in Section 3, some special cases of sets of permutations A for which $A = \mathcal{M}(A)$. In Section 4, we conclude and give some ideas to completely solve this $A = \mathcal{M}(A)$ case.

2 Definitions and notations

A **permutation** π is a bijection of $[n] = \{1, 2, \dots, n\}$ onto itself. The set of all permutations of $[n]$ is denoted $\mathcal{S}_n$. As usual we denote a permutation π of $[n]$ as

*partly supported by NSERC through an Individual Discovery Grant (Hamel)

$\pi = \pi_1 \pi_2 \dots \pi_n$. The **identity permutation** correspond to the identity bijection of $[n]$ and is denoted $\iota = 12 \dots n$. A pair (π_i, π_j) of elements of the permutation π is called an **inversion** if $\pi_i > \pi_j$ and $i < j$. The **number of inversion** of a permutation π is denoted $\text{inv}(\pi)$.

The **Kendall- τ distance**, denoted d_{KT} , counts the number of pairwise disagreements between two permutations and can be defined formally as follows: for permutations π and σ of $[n]$, we have that

$$d_{KT}(\pi, \sigma) = |\{(i, j) : i < j \text{ and } [(\pi[i] < \pi[j] \text{ and } \sigma[i] > \sigma[j]) \\ \text{or } (\pi[i] > \pi[j] \text{ and } \sigma[i] < \sigma[j])]\}|,$$

where $\pi[i]$ denote the position of integer i in permutation π . Note that we can easily computed $\text{inv}(\pi)$ as $\text{inv}(\pi) = d_{KT}(\pi, \iota)$.

The problem of finding $\mathcal{M}(A)$, the set of all medians of A under the Kendall- τ distance can be stated formally as follow:

Given $A \subseteq \mathcal{S}_n$, we want to find all permutations π^* of $[n]$ such that

$$d_{KT}(\pi^*, A) \leq d_{KT}(\pi, A), \text{ for all } \pi \in \mathcal{S}_n.$$

3 The case $A = \mathcal{M}(A)$

In this section, we give two special cases when the set of medians of A is equal to A itself. The first case (see Theorem 1) is when A is composed of a permutation and its "circular rotations". The second one (see Theorem 2) is when A consist of permutations composed of a common S_k kernel with some (or none) shared fixed points.

Theorem 1 *Let $\pi = \pi_1 \dots \pi_n$ be a permutation of $[n]$ and let $\uparrow_i \pi = \pi_{i+1} \dots \pi_n \pi_1 \dots \pi_i$. If $A = \{\uparrow_i \pi \mid 0 \leq i \leq n-1\}$ then $A = \mathcal{M}(A)$ and, $\forall i, 0 \leq i \leq n-1$, $d_{KT}(\uparrow_i \pi, A) = \frac{(n+1)n(n-1)}{6}$.*

Proof. We have that the set A contains the following permutations:

$$\begin{array}{ccccccc} \pi_1 & \pi_2 & \pi_3 & \dots & \pi_{n-2} & \pi_{n-1} & \pi_n \\ \pi_2 & \pi_3 & \pi_4 & \dots & \pi_{n-1} & \pi_n & \pi_1 \\ \pi_3 & \pi_4 & \pi_5 & \dots & \pi_n & \pi_1 & \pi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \pi_n & \pi_1 & \pi_2 & \dots & \pi_{n-3} & \pi_{n-2} & \pi_{n-1} \end{array}$$

Now to see that $d_{KT}(\uparrow_i \pi, A) = (n+1)n(n-1)/6$ just start with $\uparrow_i \pi$ for any i and counts the number of disagreements with all permutations in set A . For any ℓ ,

$1 \leq \ell \leq n-1$, any of the $n-\ell$ pairs of elements π_ℓ and $\pi_{\ell+k}$, $1 \leq k \leq n-\ell$, in $\uparrow_i \pi$ appears in the order π_ℓ before $\pi_{\ell+k}$ in all permutations except the ℓ permutations of A having respectively $\pi_\ell, \pi_{\ell+1}, \dots, \pi_{\ell+k-1}$ as its last element. So we have

$$d_{KT}(\uparrow_i \pi, A) = \sum_{\ell=1}^n (n-\ell)\ell = \frac{(n-1)n(n+1)}{6}.$$

To finish the proof, note that there are no pairs of elements that appears in the same order in all permutations of A . That means that any pairs of elements of a permutation σ will contribute with a strictly positive factor to the distance $d_{KT}(\sigma, A)$. Now, the permutations in A are the only one in S_n that have $n-1$ pairs of element (all consecutive pairs π_i, π_{i+1}) that contribute only by 1 to the distance. That means that for any other permutation $\sigma \in S_n \setminus A$, we only have $t < n-1$ pairs of elements that contribute by 1 to the distance. All the order pairs contribute by a factor $f > 1$. We can then conclude that $d_{KT}(\sigma, A) > d_{KT}(\uparrow_i \pi, A), \forall i$, which means that $A = \mathcal{M}(A)$. ■

Theorem 2 *If $A = \{12 \dots \ell \sigma \ell+k+1 \dots n \mid \sigma \text{ any permutation of } \{\ell+1, \dots, \ell+k\}\}$ for $0 \leq \ell \leq n-1$, $1 \leq k \leq n$, then $A = \mathcal{M}(A)$ and for any $\pi \in A$, $d_{KT}(\pi, A) = \frac{k(k-1)k!}{4}$. Note that $A = S_n$, is the special case when $\ell = 0$ and $k = n$.*

Proof. For any permutation $\gamma \in S_n$, $d_{KT}(\gamma, A)$ contains the distance to S_k . This distance is the total number of inversions in all permutations of $[k]$, which is $\frac{k(k-1)k!}{4}$ [10, 11]. Since for any permutation $\in A$, the other elements are fixed points, $d_{KT}(\pi, A) = \frac{k(k-1)k!}{4}$, and these element represent the elements of S_n with minimal Kendall- τ distance from A . ■

4 Conclusion and future works

Since the problem of finding medians of a set of permutation under the Kendall- τ distance is NP-complete in general, finding special sets A for which it is easy to find the medians is useful for computational purposes. In this extended abstract we have just begin to study the special case $A = \mathcal{M}(A)$ but there are still a lot of interesting ideas to investigate. The cases $A \subset \mathcal{M}(A)$, $\mathcal{M}(A) \subset A$ and $A \cap \mathcal{M}(A) = \emptyset$ are only three of some interesting future directions that needs to be evaluate.

Coming back to our present case, the two theorems presented in Section 3 gives us the basis for building further theory on the $A = \mathcal{M}(A)$ problem. If we extend Theorem 2 to a composition of basic structures (circular types, as in Theorem 1, and S_k types), we conjecture that this type of composition keeps the $A = \mathcal{M}(A)$ property. With this extension, we can rethink the permutations of A , in Theorem 2, as a composition of a S_k structure with some or none S_1 structures (the fixed points).

We observed that some special permutation sets display the $A = \mathcal{M}(A)$ property by addition of basic structures instead of composition. It would be interesting to

verify if there exist a relation between circular and S_k structures, knowing that a S_k structure can be viewed as an addition of circular components.

For now, we leave you with this question, that we plan to answer in the near future: Can we characterize all the $A = \mathcal{M}(A)$ cases?

References

- [1] N. Ailon, M. Charikar and N. Newman, *Aggregating inconsistent information: Ranking and clustering*, In Proceedings of the 37th STOC, pp.684–693, 2005.
- [2] N. Betzler, M.R. Fellows, J. Guo, R. Niedermeier and F.A. Rosamond, *Fixed-Parameter Algorithms for Kemeny Scores*, LNCS 5034, pages 60–71, 2008.
- [3] T. Biedl, F.J. Brandenburg and X. Deng, *Crossings and Permutations*, LNCS 3843, pages 1–12, 2005.
- [4] G. Blin, M. Crochemore, S. Hamel et S. Vialette, *Medians of an odd number of permutations*, Pure Mathematics and Applications, Volume 21, Issue 2, pp. 161–175, 2010.
- [5] V. Conitzer, A. Davenport and J. Kalagnanam, *Improved Bounds for Computing Kemeny Rankings*, in Proceedings of the 21st National Conference on Artificial Intelligence (AAAI), pages 620–627, 2006.
- [6] C. Dwork, R. Kumar, M. naor and D. Sivakumar, *Rank Aggregation Methods for the Web*, in proceedings of the 10th WWW, pp.613-622, 2001.
- [7] J. Kemeny, *Mathematics without numbers*, Daedalus 88 , pp. 577591, 1959.
- [8] M. Kendall, *A New Measure of Rank Correlation*, Biometrika, 30, 81-89, 1938.
- [9] C. Kenyon-Mathieu and W. Schudy, *How to rank with few errors*, In Proceedings of the 39th STOC, pp. 95-103, 2007.
- [10] M. Stern, *Aufgaben*, Jour. für reine und angew. Math., 18, pp. 100, 1838.
- [11] M. O. Terquem, *Solution d'un Problème de combinaison*, J. de Math. Pures et Appl., 3, pp. 559-560, 1838.
- [12] M. Truchon, *An Extension of the Condorcet Criterion and Kemeny Orders*, Internal Report, cahier 98-15 du Centre de Recherche en Économie et Finance Appliquées, Université Laval, 16 pages, 1998.
- [13] A. vanZuylen and D.P. Williamson, *Deterministic algorithms for rank aggregation and other ranking and clustering problems*, in Proceedings of the 5th WAOA, LNCS 4927, pp.260–273, 2007.

Functional equations for enumerating 1324-avoiding permutations

Brian Nakamura

In previous work, Noonan and Zeilberger derived functional equations that encoded the occurrences of certain patterns of length 3 and 4. In later work, this was adapted into a polynomial time algorithm for enumerating permutations with exactly r copies of a pattern (for small, fixed r and for certain small patterns). In this talk, we will discuss some additional extensions to this approach. This also leads to a new enumeration approach for the 1324-avoiding permutations.

Pattern avoidance on increasing binary trees Peter Nugent*, Lara Pudwell, Manda Riehl, ML Tlachac*

An increasing unary-binary tree is a tree graph with n nodes labeled from 1 to n , where each node has 0, 1 or 2 children, and every path from root to leaf is an increasing sequence of labels. These objects contain an associated permutation by reading left to right across the levels of the tree starting with the root. We will examine the number of both unary-binary trees and binary trees whose associated permutation avoids patterns of length 3, and how these trees are related to other combinatorial objects. For example, the numbers of increasing unary-binary trees avoiding the pattern 132 are given by the Motzkin numbers, and we present a bijection to the classical interpretation as nonintersecting chords drawn on a circle.

Increasing unary-binary trees	Sequence	A#
Total	1, 1, 3, 9, 39, 189, 1107, 7281...	A080635
$Av(123)$	1, 1, 1, 0, 0, 0, 0...	?
$Av(132) =$ $Av(132, 213) =$ $Av(132, 231) =$ $Av(132, 312) =$ $Av(132, 321)$	1, 1, 2, 3, 9, 21, 51, 127...	A001006: Motzkin numbers
$Av(213)$	1, 1, 3, 7, 22, 65, 213, 693...	A166135
$Av(231)$	1, 1, 3, 9, 36, 155, 752, 3894...	?
$Av(312)$	1, 1, 3, 8, 29, 110, 469, 2119...	?
$Av(321)$	1, 1, 3, 9, 37, 165, 834, 4515...	?
$Av(213, 231)$	1, 1, 3, 7, 21, 61, 191, 603...	A025235
$Av(213, 312)$	1, 1, 3, 6, 16, 39, 103, 269...	A188442
$Av(213, 321)$	1, 1, 3, 7, 20, 55, 157, 448...	?
$Av(231, 312) =$ $Av(312, 321)$	1, 1, 3, 8, 28, 102, 407, 1701...	?
$Av(231, 321)$	1, 1, 3, 9, 34, 134, 568, 2499...	?

Bijection from Motzkin numbers to unary-binary tree structures

Since each label in an increasing unary-binary tree is larger than the label of its parent, the only permutation of any length $n > 2$ that we can create on an increasing tree that avoids 132 is the identity permutation $1\ 2\ 3\ \dots\ n-1\ n$. Then the number of unique trees with associated permutation that avoid 132 for any tree with p nodes

Increasing binary trees	Sequence	A#
total	1,2,16,272,7936...	A000182: Tangent numbers
$Av(123)$	1, 1, 0, 0, 0...	?
$Av(132) =$ $Av(132, 213) =$ $Av(132, 231) =$ $Av(132, 321) =$ $Av(132, 321)$	1, 1, 2, 5, 14...	A000108: Catalan numbers
$Av(213)$	1, 2, 9, 52, 340...	A069271
$Av(231)$	1, 2, 14, 163, 2558...	?
$Av(312)$	1, 2, 11, 96, 1093...	?
$Av(321)$	1, 2, 14, 165, 2639...	?
$Av(213, 231)$	1, 2, 8, 40, 224...	$2 * A003645 = 2^n C_n$
$Av(213, 312)$	1, 2, 6, 20, 70...	A000984: Central binomial coefficients
$Av(213, 321)$	1, 2, 7, 26, 98...	A126223: level steps on a variant of 2-Motzkin paths
$Av(231, 312) =$ $Av(312, 321)$	1, 2, 10, 72, 632...	?
$Av(231, 321)$	1, 2, 12, 98, 940...	?

is simply equal to the number of unique unary-binary trees with p nodes, which we will prove to be the Motzkin numbers.

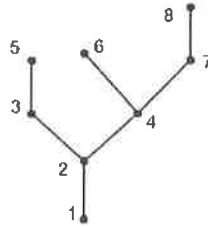


Figure 1: An unary-binary tree with 9 nodes associated with the identity permutation

The n^{th} Motzkin numbers are defined by the number of paths from $(0,0)$ to $(n,0)$ using the steps $(1,0)$, $(1,1)$ and $(-1,1)$. This is called a Motzkin n -path (Figure (2a)). Each of these paths can also be mapped to a unary-binary tree structure with n nodes (Figure (2b)). For this map ϕ , to find the corresponding path for a tree graph, perform a depth-first search starting at the root going clockwise and mark

down each time an edge is encountered for the first time, and if that edge is a left, right, or center. Any edge that has no siblings is a center, and for any edge that has siblings the first one encountered of the two will be the left, and the second will be the right. For each edge, you will take a corresponding step. Each left edge is a $(1, 1)$ step, a center edge is a $(1, 0)$ and a right is $(1, -1)$ step. This will create a graph that goes from $(0, 0)$ to $(n, 0)$, using only $(1, 0)$, $(1, 1)$, and $(1, -1)$ steps that is unique to each tree.

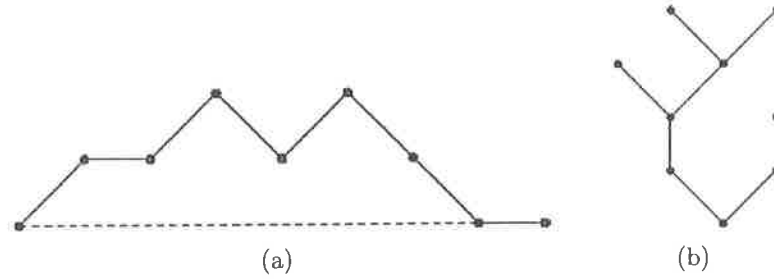


Figure 2: (a) A motzkin 8-path (b) an unary-binary tree with 9 vertices

This map is reversible, meaning a Motzkin n -path can be used to find the corresponding unary-binary tree. For example, Figure 3(a) can be translated into a string of rights, lefts and centers based on its steps and their order. Using this information, we can find Figure 3(b) to be the corresponding tree, so that $\phi^{-1}(3b) = (3a)$

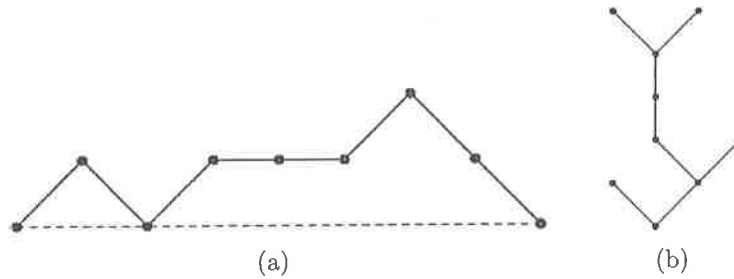


Figure 3: (a) A motzkin path of length 8 (b) the unary-binary tree associated with 3(a)

Bijection between unary-binary trees with n nodes and the classical Motzkin interpretation as nonintersecting chords drawn on $n-1$ nodes on a circle

Figure 4 shows an unary-binary tree. It is a combination of left, right, and center edges. We create a word associated to our tree by performing a depth-first search starting at the root and moving clockwise. As we first encounter an edge, we record that edge as left, center, or right to create the associated path word. In Figure 5(a)

we labeled the edges with the order we encountered them on our depth-first search.

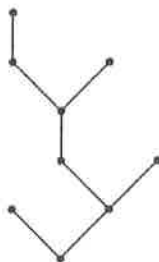


Figure 4: T_3 , an unary-binary tree with 9 nodes

Figure 5(b) shows a circle with nonintersecting chords drawn between some of the numbered nodes. By nonintersecting we mean both that the chords don't cross and that each node has zero or one chords incident to it.

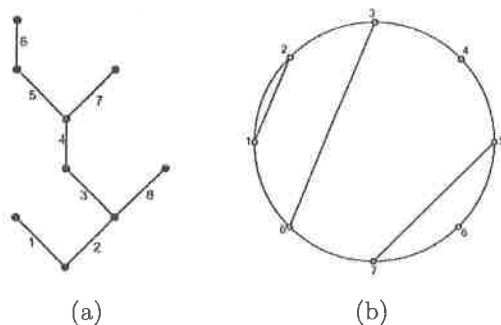


Figure 5: (a) T_3 , with edges labelled by depth-first search, with associated path word LRLCLCRR, (b) Nonintersecting chords drawn on a circle with 8 nodes

Finding a circle with nonintersecting chords from an unary-binary tree:

The circle should be drawn as to have as many nodes as the tree has edges. The left, right, and center associated path word for the tree indicate whether the nodes on the corresponding circle will have a chord. We read clockwise around the circle starting at the first numbered node (the first letter corresponds to the node 1, the second letter corresponds to node 2, etc.). The left direction indicates a chord is started from that node, a right direction indicated a chord ends at that node, and a center direction indicates that there are no chords incident to that point. Notice that the labels on a chord are a right/left pair in the tree's associated path word.

As such, it is possible to create a unique circle with nonintersecting chords from the structure of a tree.

Finding an unary-binary tree from a circle with nonintersecting chords:

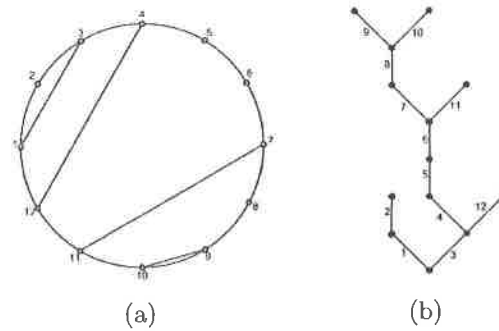


Figure 6: (a) Nonintersecting chords drawn on a circle with 12 nodes and associated path word LCRLCCLCLRRR, (b) Unary-binary tree with edges labelled by depth-first search

We can reverse the above process by starting with a circle with nonintersecting chords and obtaining a corresponding unary-binary tree. The first step is to create the left, right, and center string that will indicate order of the edges of the tree from the labelled points of the circle. To do this, read the points clockwise starting at label 1 and record which segment they represent; as mentioned above, the start of a chord indicates a left, an end of a chord indicates a right, and a point with no chord indicates a center. We construct trees by starting at the root and reading the left, right, and center string. Whenever a right is reached, do not add it to the end of the current branch as is done with the left and center pieces, but add it to a previous unit such that it creates a left/right pair. As such a uniquely structured tree can be built from a circle of nonintersecting chords.

As this is an ongoing project which we only began recently, we are continuing to work on finding proofs (preferably bijective) that the conjectured sequences in the table enumerate the trees which avoid a given pattern. Currently, we are focusing on a bijective proof that the number of increasing binary trees whose permutation avoids 213 and 321 is counted by $A126223$.

PATTERN-AVOIDING INVOLUTIONS: EXACT AND ASYMPTOTIC ENUMERATION

Miklós Bóna, Cheyne Homberger, Jay Pantone, and Vincent Vatter

Department of Mathematics
University of Florida
Gainesville, Florida USA

1. INTRODUCTION

For the past twenty-five years, there has been considerable interest in the enumeration of pattern-avoiding permutations. Much less work has been devoted to pattern-avoiding *involutions*. An involution is a permutation which is its own group-theoretic inverse. Given a basis B , we denote the set of B -avoiding involutions, by

$$\text{Av}^I(B) = \{\text{involutions } \pi : \pi \text{ avoids every } \beta \in B\}.$$

Note that $\text{Av}^I(B)$ is *not* a permutation class in general. We define the generating functions and the notion of Wilf-equivalence for such sets analogously to those of permutation classes.

The enumeration of sets of involutions avoiding a pattern β of length 3 was first considered in the seminal paper of Simion and Schmidt [13], where it was shown that there are two different enumerations (contrary to the single enumeration in the case of all permutations). The situation gets much more complicated when $|\beta| = 4$. Extending the work of Guibert [9] and Guibert, Pergola, and Pinzani [10], Jaggard [11] completed the classification of Wilf-equivalence classes of sets of involutions avoiding a pattern of length 4. Of these eight Wilf-classes, only two enumerations have been computed so far. Gessel [8] provided the enumeration of $\text{Av}^I(1234)$, while Brignall, Huczynska, and Vatter [7] counted $\text{Av}^I(2413)$.

Jaggard concluded his paper by presenting the first few terms of the enumerations of these Wilf-equivalence classes, which we show in Table 1. The order of the columns of this table is determined by the number of permutations of length 11 which avoid each pattern. Our interest in the topic of pattern-avoiding involutions was first piqued when we noticed that this ordering *is incredibly misleading* (through no fault of Jaggard's).

2. GROWTH RATES AND THE DECEPTIVENESS OF TABLE 1

To explain why the ordering of the columns of Table 1 must be incorrect, we want to look at the asymptotic, rather than exact, enumeration of such sets. It is known that principally based classes

	1324	1234	4231	2431	1342	2341	3421	2413
$ \text{Av}_5^I(\beta) $	21	21	21	24	24	25	25	24
$ \text{Av}_6^I(\beta) $	51	51	51	62	62	66	66	64
$ \text{Av}_7^I(\beta) $	126	127	128	154	156	170	173	166
$ \text{Av}_8^I(\beta) $	321	323	327	396	406	441	460	456
$ \text{Av}_9^I(\beta) $	820	835	858	992	1040	1124	1218	1234
$ \text{Av}_{10}^I(\beta) $	2160	2188	2272	2536	2714	2870	3240	3454
$ \text{Av}_{11}^I(\beta) $	5654	5798	6146	6376	7012	7273	8602	9600

Table 1: The enumerations of involutions avoiding a pattern β of length 4 for $n = 5, \dots, 11$, as presented by Jaggard [11].

have proper growth rates; however, when we restrict to involutions, the same argument does not go through. We can however, prove the following.

Proposition 2.1. *If the permutation β is sum indecomposable, then $\text{Av}^I(\beta)$ has a proper growth rate.*

Let $\text{gr}(\mathcal{C})$ denote the growth rate of the class $\mathcal{C}$. For $|\beta| = 4$ there are three possible growth rates of $\text{Av}(\beta)$. Regev [12] showed that $\text{gr}(\text{Av}(1234)) = 9$, while Bóna's work [6] shows that $\text{gr}(\text{Av}(1342)) = 8$. The final value, that of $\text{gr}(\text{Av}(1324))$, is currently unknown, although we have bounds in both directions. For the upper-bound, Bóna [5] has showed that $\text{gr}(\text{Av}(1324)) \leq (2 + \sqrt{3})^2 \approx 13.93$. The best current lower bound on $\text{gr}(\text{Av}(1324))$ is 9.47, due to Albert, Elder, Rechnitzer, Westcott, and Zabrocki [3]. Those authors also speculated, based on numerical evidence, that the true value of $\text{gr}(\text{Av}(1324))$ lies between 11 and 12.

Next we provide a relation between β -avoiding permutations and β -avoiding involutions in the case where β is a skew indecomposable involution (such as $\beta = 1324$, which is the case we want it for).

Proposition 2.2. *For every skew indecomposable involution β , we have*

$$\overline{\text{gr}}(\text{Av}^I(\beta)) \geq \sqrt{\text{gr}(\text{Av}(\beta))},$$

where $\overline{\text{gr}}(\mathcal{C}) = \limsup_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n|}$ is the upper growth rate of the class $\mathcal{C}$.

The bound from [3] and Proposition 2.2 therefore imply that

$$\overline{\text{gr}}(\text{Av}^I(1324)) > 3.07,$$

and thus the number of 1324-avoiding involutions must overtake the number of 1234-avoiding involutions at some point. Moreover, the number of 1324-avoiding involutions should overtake the number of 2413-avoiding involutions, unless $\text{gr}(\text{Av}(1324)) < 9.9$, which does not seem especially likely.

We conclude this section by updating Jaggard's table to include data up to $n = 20$ in Table 2, which was computed using Albert's PermLab package [1]. This data shows that the number of 1324-avoiding involutions first overtakes the number of 1234-avoiding involutions at $n = 18$, and does not overtake the number of 2413-avoiding involutions for $n \leq 20$. Thus by our remarks above, the ordering of the columns in Table 2 is likely still incorrect.

	2431	2341	1342	1234	1324	3421	4231	2413
$ \text{Av}_{12}^I(\beta) $	16238	18477	18322	15511	15272	22878	16716	27246
$ \text{Av}_{13}^I(\beta) $	40914	46825	47560	41835	40758	60794	46246	77132
$ \text{Av}_{14}^I(\beta) $	103954	118917	124358	113634	112280	161668	128414	221336
$ \text{Av}_{15}^I(\beta) $	262298	301734	323708	310572	304471	429752	361493	635078
$ \text{Av}_{16}^I(\beta) $	665478	766525	846766	853467	852164	1142758	1020506	1839000
$ \text{Av}_{17}^I(\beta) $	1680726	1946293	2208032	2356779	2341980	3038173	2913060	5331274
$ \text{Av}_{18}^I(\beta) $	4260262	4944614	5777330	6536382	6640755	8078606	8335405	15555586
$ \text{Av}_{19}^I(\beta) $	10766470	12557685	15082372	18199284	18460066	21479469	24067930	45465412
$ \text{Av}_{20}^I(\beta) $	27274444	31900554	39469786	50852019	52915999	57113888	69646035	133517130
growth rate	?	≈ 2.54	≈ 2.62	$\frac{3}{\text{Regev [12]}}$	$> 3.07, < 4.84$	?	?	≈ 3.15 BHV [7]
OEIS	A230551	A230552	A230553	A001006	A230554	A230555	A230556	A121704

Table 2: The enumerations of involutions avoiding a pattern of length 4 for $n = 12, \dots, 20$, with columns sorted according to the number of involutions of length 20 avoiding the given pattern. Note that the bounds for 1324 are actually bounds of the upper growth rate.

3. COUNTING 321-AVOIDING SIMPLE INVOLUTIONS

In both of the sets we enumerate, (nearly all of) the simple involutions avoid 123. In fact, we need to refine this enumeration to count these permutations by their number of fixed points, $\text{fp}(\sigma)$, number of left-to-right minima, $\text{lrmin}(\sigma)$, and number of right-to-left maxima, $\text{rlmax}(\sigma)$. Note that every entry of a 123-avoiding permutation is either a left-to-right minimum or a right-to-left maximum. Moreover, no entry of a *simple* 123-avoiding permutation of length at least four can be both a left-to-right minimum and a right-to-left maximum (because then the permutation would be skew decomposable). Thus the generating functions we are interested in are

$$\hat{s}^{(i)}(u, v) = \sum_{\substack{\text{simple } \sigma \in \text{Av}^I(123) \\ \text{with } \text{fp}(\sigma) = i}} u^{\text{lrmin}(\sigma)} v^{\text{rlmax}(\sigma)}.$$

These generating functions are found by an adaptation of the *staircase decomposition* introduced by Albert, Atkinson, Brignall, Ruškuc, Smith, and West [2] as part of the study of subclasses of $\text{Av}(321)$.

4. ENUMERATION OF $\text{Av}^I(1342)$

In their investigation of substitution closures of principally based classes, Atkinson, Ruškuc, and Smith [4] established that

$$\langle \text{Av}(123) \rangle = \text{Av}(24153, 25314, 31524, 41352, 246135, 415263).$$

By inspection, it is clear that each of the basis elements of $\langle \text{Av}(123) \rangle$ contains either 1342 or 1423, and thus $\text{Av}(1342, 1423) \subseteq \langle \text{Av}(123) \rangle$. This gives us the structural result we need.

Proposition 4.1. *The simple permutations of $\text{Av}(1342, 1423)$ are precisely the same as the simple permutations of $\text{Av}(123)$.*

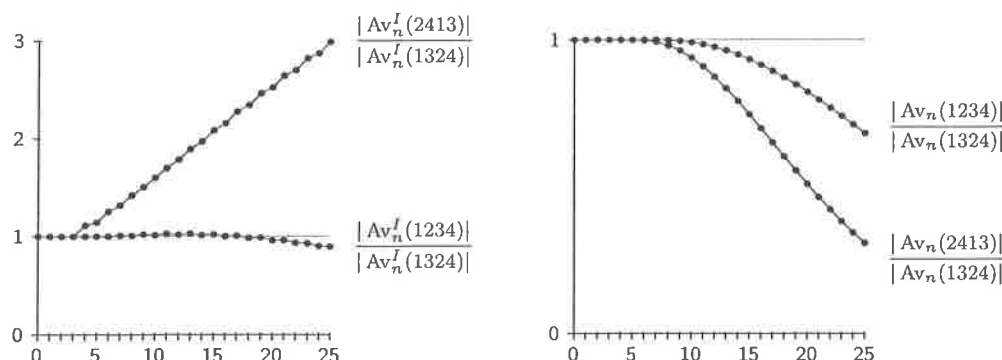


Figure 3: The number of 1324-avoiding involutions and permutations compared to the number of 1234- and 2413-avoiding involutions and 1234- and 1342-avoiding permutations for $n = 0, \dots, 25$.

By considering which inflations of each simple permutation are allowed, we find the generating function for this set:

$$g(x) = \frac{x(1 - 2x + x^2 + \sqrt{1 - 6x^2 + x^4})}{2(1 - 3x + x^2)}.$$

It can then be computed that the growth rate of involutions avoiding 1342 is 1 plus the golden ratio, approximately 2.62.

5. ENUMERATION OF $\text{Av}^I(2341)$

Through a rather more involved argument than in the previous section, we prove the following theorem regarding the simple involutions contained in $\text{Av}^I(2341)$.

Theorem 5.1. *The simple 2341-avoiding involutions consist exactly of the permutation 5274163 along with the simple involutions of the class $\text{Av}^I(123)$.*

From this, the minimal polynomial of the generating function can be computed, yielding a growth rate of approximately 2.54.

6. INVOLUTIONS AVOIDING 1324 REVISITED

As stated in the introduction, we were initially interested in pattern-avoiding involutions because we noticed that $\text{gr}(\text{Av}^I(1324)) > \text{gr}(\text{Av}^I(1234))$, but that the numbers in Table 1 did not obey this relationship. The ratio between $|\text{Av}_n^I(1324)|$ and $|\text{Av}_n^I(1234)|$ is plotted on the left of Figure 3. Here we see that for large enough n , this ratio does indeed go below 1 (and of course we know that it goes to 0).

One of the most intriguing observations we have found thus far is the relationship between 1324-

and 2413-avoiding involutions. Recall that if

$$\frac{|\text{Av}_n^I(2413)|}{|\text{Av}_n^I(1324)|}$$

does not go to 0 then

$$\text{gr}(\text{Av}(1324)) \leq (\sqrt{2} + \sqrt{3})^2 < 9.9.$$

This implication seems very unlikely as it would go against the numerical evidence found by Albert, Elder, Rechnitzer, Westcott, and Zabrocki [3], which suggests that $\text{gr}(\text{Av}(1324))$ lies between 11 and 12. However, this ratio is also plotted on the left of Figure 3, and this plot certainly does not look like the plot of a sequence tending to 0. For comparison, the plots on the right of Figure 3 show the same ratios for permutations rather than involutions. These plots both tend to 0, as we know they will.

REFERENCES

- [1] ALBERT, M. H. PermLab: Software for permutation patterns. Available online at <http://www.cs.otago.ac.nz/PermLab>, 2012.
- [2] ALBERT, M. H., ATKINSON, M. D., BRIGNALL, R., RUŠKUC, N., SMITH, R., AND WEST, J. Growth rates for subclasses of $\text{Av}(321)$. *Electron. J. Combin.* 17 (2010), Paper 141, 16 pp.
- [3] ALBERT, M. H., ELDER, M., RECHNITZER, A., WESTCOTT, P., AND ZABROCKI, M. On the Stanley-Wilf limit of 4231-avoiding permutations and a conjecture of Arratia. *Adv. in Appl. Math.* 36, 2 (2006), 96–105.
- [4] ATKINSON, M. D., RUŠKUC, N., AND SMITH, R. Substitution-closed pattern classes. *J. Combin. Theory Ser. A* 118 (2011), 317–340.
- [5] BÓNA, M. A new upper bound for 1324-avoiding permutations. arXiv:1207.2379v1 [math.CO].
- [6] BÓNA, M. Exact enumeration of 1342-avoiding permutations: a close link with labeled trees and planar maps. *J. Combin. Theory Ser. A* 80, 2 (1997), 257–272.
- [7] BRIGNALL, R., HUCZYNSKA, S., AND VATTER, V. Simple permutations and algebraic generating functions. *J. Combin. Theory Ser. A* 115, 3 (2008), 423–441.
- [8] GESSEL, I. M. Symmetric functions and P -recursiveness. *J. Combin. Theory Ser. A* 53, 2 (1990), 257–285.
- [9] GUIBERT, O. *Combinatoire des permutations à motifs exclus, en liaison avec mots, cartes planaires et tableaux de Young*. PhD thesis, LaBRI, Université Bordeaux 1, 1995.
- [10] GUIBERT, O., PERGOLA, E., AND PINZANI, R. Vexillary involutions are enumerated by motzkin numbers. *Ann. Comb.* 5, 2 (2001), 153–147.
- [11] JAGGARD, A. D. Prefix exchanging and pattern avoidance by involutions. *Electron. J. Combin.* 9, 2 (2002/03), Paper 16, 24 pp.
- [12] REGEV, A. Asymptotic values for degrees associated with strips of Young diagrams. *Adv. in Math.* 41, 2 (1981), 115–136.
- [13] SIMION, R., AND SCHMIDT, F. W. Restricted permutations. *European J. Combin.* 6, 4 (1985), 383–406.

AUTOMATIC ENUMERATION AND RANDOM GENERATION FOR PATTERN-AVOIDING PERMUTATION CLASSES

A. PIERROT, joint work with F. BASSINO, M. BOUVEL, C. PIVOTEAU, D. ROSSIN

Abstract. This article presents an algorithm which computes both a system of equations for the generating function, and a random sampler for pattern-avoiding permutation classes. The algorithm is general: the permutation class is taken as input by the algorithm (the class being given by its set of forbidden patterns). However the algorithm does not compute its result for any permutation class: when the class contains an infinite number of simple permutations (this condition being tested by the algorithm itself), the algorithm stops.

The main tool is the substitution-decomposition of permutations. It is used to obtain a *combinatorial specification*, that is, an unambiguous system of combinatorial equations that describe recursively the permutations of the class. The obtained specification yields by routine algorithms a system of equations satisfied by the generating function of the class, and also a uniform random sampler of permutations in the class.

1. INTRODUCTION

Since the definition of the pattern relation among permutations by Knuth in the 70's [Knu73], the study of permutation patterns and permutation classes in combinatorics has been a quickly growing research field, and is now well-established. Most of the research done in this domain concerns *enumeration* questions on permutation classes (see [BM03, Eli04, KM03] and their references among many others). Most articles are focused on a given class $\mathcal{C} = Av(B)$ where the basis B of excluded patterns characterizing $\mathcal{C}$ is finite, explicit, and in most cases contains only patterns of size 3 or 4. Another line of research on permutation classes has been emerging for almost a decade: it is interested in properties or results that are less precise but apply to *families* of permutation classes that are as wide as possible. This second point of view is not purely combinatorial but instead is intimately linked with algorithms. Indeed, when stating general structural results on families of permutation classes, it is natural to associate to an existential theorem an algorithm that *tests* whether a class given in input falls into the family of classes covered by the theorem, and in this case to *compute* the result whose existence is guaranteed by the theorem.

Certainly the best illustration of this paradigm that can be found in the literature is the result of Albert and Atkinson [AA05], stating that every permutation class containing a finite number of simple permutations has a finite basis and an algebraic generating function, and its developments by Brignall *et al.* in [BHV08b, BHV08a, BRV08].

In this theorem, as well as in other results obtained in this field [AAK03, Bri10, Vat08], it appears that *simple permutations* play a crucial role, in particular as building blocks for permutations using substitution.

In addition more could and should be done on the algorithmization of finding structure in permutation classes. This article is resolutely turned towards algorithmic considerations. It provides efficient algorithms that not only *test* whether there is a finite number of simple permutations in a permutation class $\mathcal{C}$, but that also *compute* in this case an algebraic system

of equations satisfied by the generating function of $\mathcal{C}$. More precisely, we compute a combinatorial specification for $\mathcal{C}$. By a *combinatorial specification* of a class (see [FS09]), we mean an unambiguous system of combinatorial equations that describes recursively the permutations of $\mathcal{C}$ using only combinatorial constructors (disjoint union, cartesian product, sequence, ...) and permutations of size 1.

We believe that our purpose of obtaining algorithmically combinatorial specifications of permutation classes is of interest *per se* but also because it then allows to obtain by routine algorithms a positive system of equations satisfied by the generating function of $\mathcal{C}$ and a uniform random sampler of permutations in $\mathcal{C}$, using the methods of [FS09] and [DFLS04] respectively.

Our work takes its root in work of Albert and Atkinson [AA05] and of Brignall *et al.* in [BHV08b, BHV08a, BRV08], which give existence results (with somehow constructive proofs) for algebraic systems and a decidability result for the finiteness of the number of simple permutations. The main interest of our work is that it is fully algorithmic. Moreover, we conducted a deep study to design efficient algorithms. We studied the poset of simple permutations, the rational language corresponding to pin-permutations (a particular kind of simple permutations) and we also introduced new concepts such as sets of permutations with both excluded and mandatory patterns.

The results presented here bear theoretical importance, and at the same time we believe they will have practical applications for the *Permutation Patterns* community.

2. ALGORITHM

We now describe the different steps of our algorithmic chain from the basis of a permutation class $\mathcal{C}$ to a positive system of equations satisfied by the generating function of $\mathcal{C}$ and a uniform random sampler of permutations in $\mathcal{C}$, which is illustrated in Figure 1. This algorithmic chain is described in details in [Pie13].

In the following, we only consider classes whose basis B is given explicitly, and is finite. This does not cover the whole range of permutation classes, but it is a reasonable assumption when dealing with *algorithms* on permutation classes, that take a finite description of a permutation class in input. Moreover, the classes that are fully handled by our algorithmic process are those containing a finite number of simple permutations, and by [AA05], such classes have necessarily finite bases.

First, we should ensure that $\mathcal{C}$ falls into the set of permutation classes we can handle, i.e. ensure that $\mathcal{C}$ contains a finite number of simple permutations. Following the line opened by [BRV08] which proved that this question is decidable, we give an algorithm for this first step in [BBPR13]: In [BRV08], new objects are introduced, most importantly, pin-permutations. Making use of the detailed study of pin-permutations in [BBR11], we provide in [BBPR13] an efficient algorithm for this decision problem. In the case where the class is substitution-closed, the algorithm is even more efficient (see [BBPR10]).

Second, when finite, we compute the set $\mathcal{S}_{\mathcal{C}}$ of simple permutations in $\mathcal{C}$. An algorithm for this second step is described in [PR12], and its complexity analyzed. It should be noticed that the complexity of this algorithm also depends on the size of its output, namely on $|\mathcal{S}_{\mathcal{C}}|$ and on $\max\{|\pi| : \pi \in \mathcal{S}_{\mathcal{C}}\}$. The algorithm is again more efficient when the class is substitution-closed.

Third, from B and $\mathcal{S}_{\mathcal{C}}$, we compute a (possibly ambiguous) system of combinatorial equations describing the permutations of $\mathcal{C}$, turning the constructive proof of [AA05] into an actual algorithm.

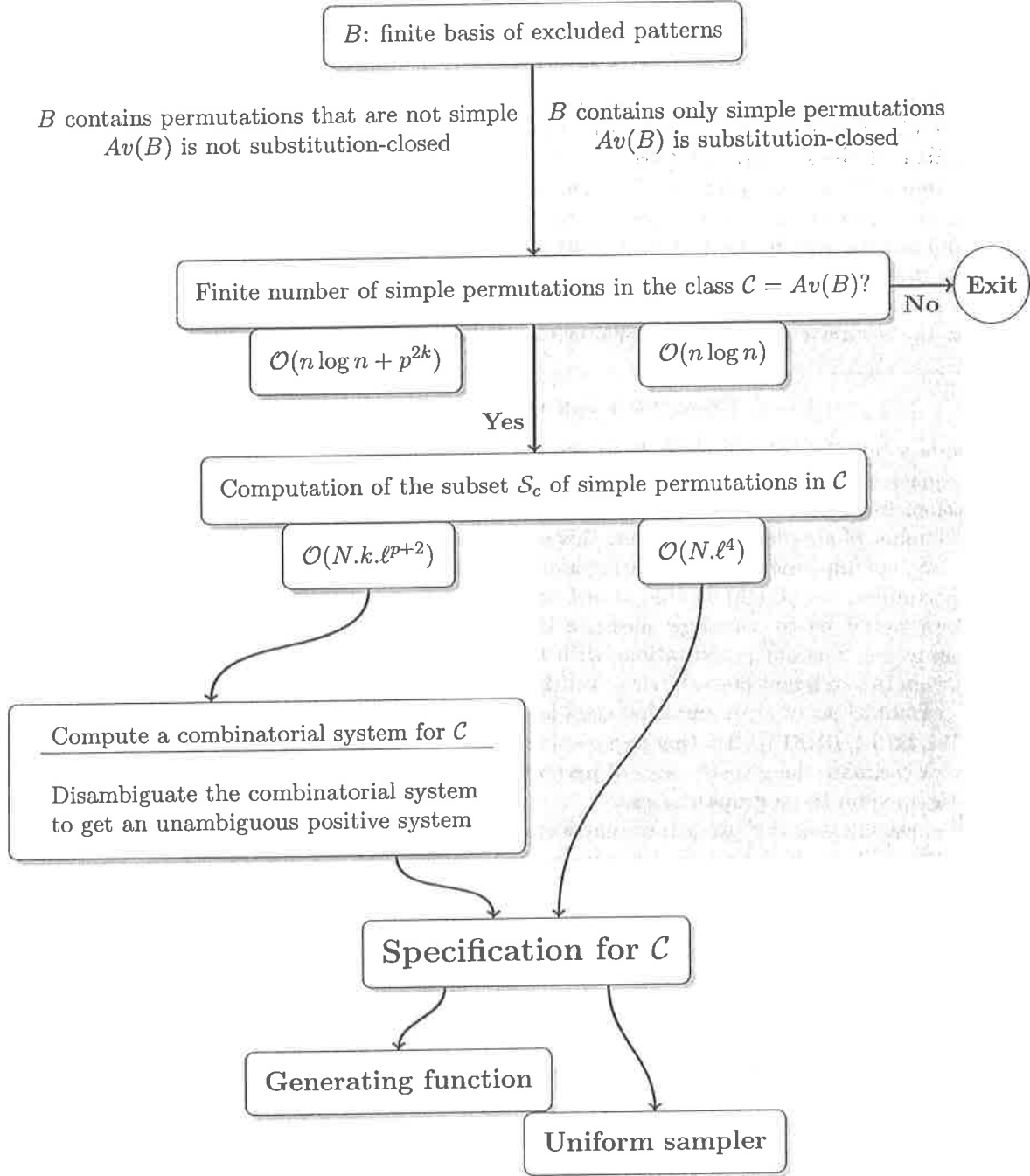


FIGURE 1. Automatic process from the basis of a permutation class to generating function and uniform sampler. The complexities are given w.r.t. $n = \sum_{\beta \in B} |\beta|$, $k = \#B$, $p = \max\{|\beta| : \beta \in B\}$, $N = \#\mathcal{S}_\mathcal{C}$ and $\ell = \max\{|\pi| : \pi \in \mathcal{S}_\mathcal{C}\}$ where $\mathcal{S}_\mathcal{C}$ is the set of simple permutations in $\mathcal{C}$.

Finally, we transform this (possibly ambiguous) system into an unambiguous combinatorial specification for $\mathcal{C}$. We describe in the abstract [BBP⁺12] an algorithm for these last two steps, whose complexity is still to be analyzed. A long version is in preparation [BBP⁺14].

Combining these four steps provides an algorithm to obtain from a basis B of excluded patterns a combinatorial specification for the permutation class $\mathcal{C} = Av(B)$ when $\mathcal{C}$ contains a finite number of simple permutations. We are not only convinced of the importance of this result from a theoretical point of view, but also (and maybe more importantly) we are confident that it will be of practical use to the *Permutation Patterns* community. Indeed, from a combinatorial specification, it is of course possible with the methodology of [FS09] to immediately deduce a system of equations for the generating function of $\mathcal{C}$. Moreover, the specification can be translated directly into a uniform random sampler of permutations in $\mathcal{C}$, using either the recursive method [FZVC94] or the Boltzmann method [DFLS04].

3. CONCLUSION AND PERSPECTIVES

We provide a full algorithmic chain from the basis of a permutation class $\mathcal{C}$ to a positive system of equations satisfied by the generating function of $\mathcal{C}$ and a uniform random sampler of permutations in $\mathcal{C}$ (see Figure 1). This procedure fail to compute its result when $\mathcal{C}$ contains an infinite number of simple permutations, this condition being tested algorithmically.

With a complete implementation of the algorithmic chain from B to the specification and of the uniform sampler, one should be able to test conjectures on and study permutation classes. One direction would be to somehow measure the randomness of permutations in a given class, by comparing random permutations with random permutations in a class, or random permutations in two different classes, w.r.t. well-known statistics on permutations. The study of random permutations in a permutation class is a new topic which seems promising [ML10, AM14, MP14, DP14, HRS14]. Another perspective would be to use the specifications obtained to compute or estimate the growth rates of permutation classes, to provide improvements on the known bounds on these growth rates.

However, a weakness of our procedure that we must acknowledge is that it fails to be completely general. Although the method is generic and algorithmic, the classes that are fully handled by the algorithmic process are those containing a finite number of simple permutations. By [AA05], such classes have finite basis, but they also have an *algebraic* generating function. Of course, this is not the case for every permutation class. We may wonder how restrictive this framework is, depending on which problems are studied. First, does it often happen that a permutation class contains finitely many simple permutations? From the work of [Vat10] it seems that a majority of permutation classes are not algebraic, thus do not contain finitely many simple permutations. But to properly express what *often* means, a probability distribution on permutation classes should be defined, which is a direction of research yet to be explored. Second, we may want to describe some problems (maybe like the distribution of some statistics) for which algebraic permutation classes are representative of all permutation classes.

To enlarge the framework of application of our algorithm, we could explore the possibility of extending it to permutation classes that contain an infinite number of simple permutations, but that are finitely described (like the family of oscillations of [BRV08] for instance). With such an improvement, more classes would enter our framework, but it would be hard to leave the algebraic case. This is however a promising direction for the construction of uniform random samplers for such permutation classes.

REFERENCES

- [AA05] M. Albert and M. Atkinson. Simple permutations and pattern restricted permutations. *Discrete Mathematics*, 300(1–3):1–15, 2005.
- [AAK03] M. Albert, M. Atkinson, and M. Klazar. The enumeration of simple permutations. *Journal of Integer Sequences*, 6, 2003.
- [AM14] M. Atapour and N. Madras. Large deviations and ratio limit theorems for pattern-avoiding permutations. *Combinatorics, Probability & Computing*, 23(2):161–200, 2014.
- [BBP⁺12] F. Bassino, M. Bouvel, A. Pierrot, C. Pivoteau, and D. Rossin. Combinatorial specification of permutation classes. In *Proceedings of FPSAC 2012, Nagoya, Japan*, volume AR of *DMTCS proceedings*, pages 781–792, 2012.
- [BBP⁺14] F. Bassino, M. Bouvel, A. Pierrot, C. Pivoteau, and D. Rossin. An algorithm computing combinatorial specifications of permutation classes. In preparation, 2014.
- [BBPR10] F. Bassino, M. Bouvel, A. Pierrot, and D. Rossin. Deciding the finiteness of the number of simple permutations contained in a wreath-closed class is polynomial. *Pure Mathematics and Applications*, 21(2):119–135, 2010.
- [BBPR13] F. Bassino, M. Bouvel, A. Pierrot, and D. Rossin. An algorithm for deciding the finiteness of the number of simple permutations in permutation classes. 2013.
- [BBR11] F. Bassino, M. Bouvel, and D. Rossin. Enumeration of pin-permutations. *Electronic Journal of Combinatorics*, 18, 2011.
- [BHV08a] R. Brignall, S. Huczynska, and V. Vatter. Decomposing simple permutations, with enumerative consequences. *Combinatorica*, 28(4):385–400, jul 2008.
- [BHV08b] R. Brignall, S. Huczynska, and V. Vatter. Simple permutations and algebraic generating functions. *Journal of Combinatorial Theory, Series A*, 115(3):423–441, 2008.
- [BM03] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: Generating trees with two labels. *Electronic Journal of Combinatorics*, 9(2), 2003.
- [Bri10] R. Brignall. A survey of simple permutations. In *Permutation Patterns, St Andrews 2007*, volume 376 of *London Math. Society Lecture Note Series*, pages 41–65. Cambridge Univ. Press, 2010.
- [BRV08] R. Brignall, N. Ruškuc, and V. Vatter. Simple permutations: decidability and unavoidable substructures. *Theoret. Comput. Sci.*, 391(1–2):150–163, 2008.
- [DFLS04] P. Duchon, P. Flajolet, G. Louchard, and G. Schaeffer. Boltzmann samplers for the random generation of combinatorial structures. *Combinatorics, Probability and Computing*, 13(4–5):577–625, 2004.
- [DP14] T. Dokos and I. Pak. The expected shape of random doubly alternating baxter permutations. *Preprint arXiv:1401.0770*, 2014.
- [Eli04] S. Elizalde. *Statistics on pattern-avoiding permutations*. PhD thesis, MIT, 2004.
- [FS09] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge Univ. Press, 2009.
- [FZVC94] P. Flajolet, P. Zimmerman, and B. Van Cutsem. A calculus for the random generation of labelled combinatorial structures. *Theoretical Computer Science*, 132(1–2):1–35, 1994.
- [HRS14] C. Hoffman, D. Rizzolo, and E. Slivken. Pattern avoiding permutations and brownian excursion. *Preprint arXiv:1406.5156*, 2014.
- [KM03] S. Kitaev and T. Mansour. A survey on certain pattern problems. available at <http://www.ru.is/kennarar/sergey/publications.html>, 2003.
- [Knu73] D. Knuth. *Fundamental Algorithms*, volume 1 of *The Art of Computer Programming*. Addison-Wesley, Reading MA, 3rd edition, 1973.
- [ML10] N. Madras and H. Liu. Random pattern-avoiding permutations. *Algorithmic Probability and Combinatorics, AMS, Providence, RI*, pages 173–194, 2010.
- [MP14] S. Miner and I. Pak. The shape of random pattern-avoiding permutations. *Advances in Applied Mathematics*, 55:86–130, 2014.
- [Pie13] A. Pierrot. *Combinatorial and algorithmic results on permutation classes*. PhD thesis, 2013.
- [PR12] A. Pierrot and D. Rossin. Simple permutation poset. 2012.
- [Vat08] V. Vatter. Enumeration schemes for restricted permutations. *Combinatorics, Probability and Computing*, 17(1):137–159, 2008.
- [Vat10] V. Vatter. Permutation classes of every growth rate above 2.48188. *Mathematika*, 56:182–192, 2010.

Ascent sequences avoiding pairs of patterns

Lara K. Pudwell*

Department of Mathematics and Computer Science
Valparaiso University
faculty.valpo.edu/lpudwell
Lara.Pudwell@valpo.edu

Andrew M. Baxter

Mathematics Department
Pennsylvania State University
www.math.rutgers.edu/~baxter/
baxter@math.psu.edu

Given an integer string $x_1 \cdots x_n$, an *ascent* is position j such that $x_j < x_{j+1}$. Write $\text{asc}(x_1 \cdots x_n)$ for the number of ascents in $x_1 \cdots x_n$. An *ascent sequence* $x_1 \cdots x_n$ is a sequence of non-negative integers such that

1. $x_1 = 0$, and
2. for $1 < i \leq n$, $x_i \leq \text{asc}(x_1 \cdots x_{i-1}) + 1$.

For example, 01234, 0120102, and 01013 are all ascent sequences, while 01024 is not since $\text{asc}(0102) = 2$. Ascent sequences have been an increasingly frequent topic of study since Bousquet-Mélou, Claesson, Dukes, and Kitaev related them to $(2+2)$ -free posets and enumerated the total number of ascent sequences [1]. Since then, various authors have connected ascent sequences to a number of other combinatorial objects [5, 6, 7, 9]. The number of ascent sequences of length n is given by the Fishburn numbers, OEIS sequence A022493.

Given a string of integers $x = x_1 \cdots x_n$, the *reduction* of x , denoted $\text{red}(x)$ is the string obtained by replacing the i th smallest digits of x with $i - 1$. For example, $\text{red}(273772) = 021220$. A *pattern* is merely a reduced string. We say that $x = x_1 \cdots x_n$ *contains* $p = p_1 \cdots p_k$ if there exists a subsequence of x order-isomorphic to p , i.e., there exist indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

Pattern σ	$\{a_\sigma(n)\}_{n \geq 1}$	OEIS	Formula
001 010 011 012	1, 2, 4, 8, 16, 32, ...	A000079	2^{n-1}
102 0102 0112	1, 2, 5, 14, 41, 122, ...	A007051	$(3^{n-1} + 1)/2$
101 021 0101	1, 2, 5, 14, 42, 132, ...	A000108	$\frac{1}{n+1} \binom{2n}{n}$

Table 1: Initial results for pattern-avoiding ascent sequences

such that $\text{red}(x_{i_1}x_{i_2} \cdots x_{i_k}) = p$. This is analogous to the classical definition of patterns for permutations, but here patterns may contain repeated digits, and patterns are normalized so that their smallest digit is 0 rather than 1. We write $a_B(n)$ for the number of ascent sequences of length n avoiding all patterns in list B .

Pattern avoidance in ascent sequence was first studied by Duncan and Steingrímsson [8]. Their main results are given in Table 1. Duncan and Steingrímsson also conjectured relationships between sequences avoiding 201, 210, 0123, 0021, or 1012 and other entries in the Online Encyclopedia of Integer Sequences [11].

Mansour and Shattuck [10] later computed the number of sequences avoiding 1012 or 0123 and showed that certain statistics on 0012-avoiding ascent sequences are equidistributed with other statistics on the set of 132-avoiding permutations. Chen, Dai, Dokos, Dwyer, and Sagan [4] proved Duncan and Steingrímsson's conjecture that the ascent statistic and the right-to-left minima statistic are each equidistributed over 021-avoiding ascent sequences and 132-avoiding permutations. A recent result of Callan [2] provides a bijection between 021-avoiding ascent sequences and Dyck paths which preserves several interesting statistics. Callan, Mansour, and Shattuck also identified the complete equivalence class of pairs of length-4 patterns such that $a_{\sigma,\tau}(n)$ is given by the Catalan numbers in [3].

We extend the work on pattern avoiding-ascent sequences by considering ascent sequences that avoid a pair of patterns of length 3 simultaneously.

Patterns B	$\{a_B(n)\}_{n \geq 1}$	OEIS	Formula
010,021	1, 2, 4, 8, 16, 32, 64, ...	A000079	2^{n-1}
101,201	1, 2, 5, 14, 42, 132, 429, ...	A000108	C_n
101,210	1, 2, 5, 14, 41, 122, 365, ...	A007051	$(3^{n-1} + 1)/2$
000,012	1, 2, 3, 3, 0, 0, 0, ...	trivial	
000,011	1, 2, 3, 4, 5, 6, 7, ...	A000027	n
000,001	1, 2, 3, 5, 8, 13, 21, ...	A000045	F_{n+1}
011,100	1, 2, 4, 7, 11, 16, 22, ...	A000124	$n(n-1)/2 + 1$
001,100	1, 2, 4, 7, 12, 20, 33, ...	A000071	$F_{n+2} - 1$
001,210	1, 2, 4, 8, 15, 26, 42, ...	A000125	$\binom{n}{3} + n$
000,101	1, 2, 4, 9, 21, 51, 127, ...	A001006	M_n
100,101	1, 2, 5, 13, 35, 97, 275, ...	A025242	
021,102	1, 2, 5, 13, 32, 74, 163, ...	A116702	$3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$
102,120	1, 2, 5, 13, 33, 81, 193, ...	A005183	$(n-1)2^{n-2} + 1$
101,120	1, 2, 5, 13, 33, 82, 202, ...	A116703	
101,110	1, 2, 5, 13, 89, 233, 610, ...	A001519	F_{2n-1}
201,210	1, 2, 5, 15, 51, 188, 731, ...	A007317	$\sum_{k=0}^{n-1} \binom{n-1}{k} C_k$

Table 2: Ascent sequences avoiding a pair of patterns of length 3

There are 13 patterns of length 3 (000, 001, 010, 100, 011, 101, 110, and the six permutations), and thus there are 78 pairs of patterns. Computation shows that these pairs produce at least 35 different avoidance sequences, 19 of which are new to OEIS.

For the 16 remaining sequences, we prove that the enumeration conjectured by OEIS is correct and, when possible, provide a bijection with appropriate other combinatorial objects. Table 2 lists a representative pattern set and formula for each of these 16 sequences, where F_n is the n th Fibonacci number (with the convention $F_0 = 0$, $F_1 = 1$), M_n is the n th Motzkin number, and C_n is the n th Catalan number. In this talk we use arguments ranging from elementary counting to bijections with Dyck paths to generating trees and highlight selected proofs from the list of 16 results in Table 2.

References

- [1] M. Bousquet-Mélou, A. Claesson, M. Dukes and S. Kitaev. $(2 + 2)$ -free posets, ascent sequences and pattern avoiding permutations, *J. Comb. Theory Ser. A* **117**(7) (2010), 884–909.
- [2] D. Callan. Another bijection for 021-avoiding ascent sequences. ArXiv e-prints, February 2014.
- [3] D. Callan, T. Mansour, and M. Shattuck. Restricted ascent sequences and Catalan numbers. ArXiv e-prints, March 2014.
- [4] W. Chen, A. Dai, T. Dokos, T. Dwyer, and B. Sagan. On 021-avoiding ascent sequences. *Electronic J. Combin.* **20**(1) (2013), #P76 (6pp.).
- [5] A. Claesson and S. Linusson. $n!$ matchings, $n!$ posets, *Proc. Am. Math. Soc.* **139** (2011), 435–449.
- [6] M. Dukes, J. Remmel, S. Kitaev and E. Steingrímsson. Enumerating $(2 + 2)$ -free posets by indistinguishable elements, *J. Comb.* **2**(1) (2011), 139–163.
- [7] M. Dukes and R. Parviainen, Ascent sequences and upper triangular matrices containing non-negative integers. *Electronic J. Combin.* **17**(1) (2010), #R53 (16pp).
- [8] P. Duncan and E. Steingrímsson. Pattern avoidance in ascent sequences. *Electronic J. Combin.* **18**(1) (2011), #P226 (17pp).
- [9] S. Kitaev and J. Remmel. Enumerating $(2 + 2)$ -free posets by the number of minimal elements and other statistics, *Discrete Applied Mathematics* **159** (2011), 2098–2108.
- [10] T. Mansour and M. Shattuck. Some enumerative results related to ascent sequences. *Discrete Mathematics* **315-316** (2014), 29–41.
- [11] OEIS Foundation Inc. (2013), The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>.

Pattern avoidance on k -ary heaps

Manda Riehl

A k -ary heap is a labelled tree with n nodes with two properties: a) it is a complete k -ary tree: all the levels of the tree are fully filled except possibly the last level, and the nodes on that level are filled from left to right, and b) the nodes are labelled with 1 to n and every path from root to leaf forms an increasing sequence of labels. For each k -ary heap, we associate a permutation by reading the labels on the heap from left to right by levels, starting at the root. We present several results on pattern avoidance in binary heaps, as well as their generalizations to k -ary heaps, their generalizations to k -ary forests of heaps, and also present several conjectures for further study. In particular we will present a bijection between forests of t heaps with v nodes each that avoid 132 and the paths starting at the origin and ending at (t, vt) using left and up steps which stay under the line $y = vx$. We will also present our proof that the number of binary heaps that avoid 231 and 312 is given by the Narayana-Zidek-Capell numbers by showing they follow the same recurrence relation.

Pattern Avoidance on Binary Heaps

Binary Heaps Avoiding:	Sequence	A#
$\emptyset$ (Total num- ber)	1, 1, 2, 3, 8, 20, 80, 210, 896 ...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0 ...	
132	1, 1, 1, 1, 1, 1, 1 ...	A000012
213	1, 1, 2, 2, 5, 5, 14, 14, 42 ...	A208355 (obvs. A000108)
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	?
321	1, 1, 2, 3, 7, 16, 45, 111, 318 ...	?
$\{213, 231\}$ = $\{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16 ...	A016116
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11 ...	?
$\{231, 312\}$ = $\{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84 ...	A002083

This is joint work with Derek Levin, Lara Pudwell and Andrew Sandberg.

A reciprocity method for computing generating functions for words which avoid consecutive patterns according to the number of descents

Jeffrey Remmel

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
jremmel@ucsd.edu

Luvreet Sangha

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
lsangha@ucsd.edu

Let $\mathbb{P} = \{1, 2, \dots\}$ denote the set of positive integers and for any $k \in \mathbb{P}$, let $[k] = \{1, \dots, k\}$. We let $\mathbb{P}^*$ ($[k]^*$) denote the set of all words over the alphabet $\mathbb{P}$ ($[k]$). Given words $u, v \in \mathbb{P}^*$, we let $|w|$ denote the length of w , $umch(w)$ denote the number of u -matches in w and $eumch(w)$ denote the number of exact u -matches in w . If $w = w_1 \dots w_n$, we let $des(w) = |\{i : w_i > w_{i+1}\}|$ and $z^w = z_{w_1} z_{w_2} \dots z_{w_n}$. Given $i, j \in \mathbb{P}$, we let $ij(w)$ denote the number of factors of w of the form ij .

Let $k \geq 2$ and u be any word in $[k]^*$ such that $des(u) = 1$. We show that how to use the reciprocity method of Jones and Remmel [1] to compute exact formulas for the generating functions

$$\begin{aligned} N^k(x, t) &= \sum_{w \in [k]^*, umch(w)=0} x^{des(w)+1} t^{|w|}, & N^{\mathbb{P}}(x, t) &= \sum_{w \in \mathbb{P}^*, umch(w)=0} x^{des(w)+1} t^{|w|}, \\ EN^k(x, t) &= \sum_{w \in [k]^*, eumch(w)=0} x^{des(w)+1} t^{|w|}, \text{ and } & EN^{\mathbb{P}}(x, t) &= \sum_{w \in \mathbb{P}^*, eumch(w)=0} x^{des(w)+1} t^{|w|}. \end{aligned}$$

Our method relies on the fact that we can use the homomorphism method to compute the generating functions

$$\sum_{w \in \mathbb{P}^*} t^{|w|} z^w \prod_{i > j} x_{ij}^{ij(w)}, \sum_{w \in \mathbb{P}^*} t^{|w|} z^w \prod_{i < j} x_{ij}^{ij(w)}, \text{ and } \sum_{w = w_1 \leq w_2 \leq \dots \leq w_n \in \mathbb{P}^*} t^{|w|} z^w \prod_i x_i^{ii(w)}.$$

For example, given a set $S \subseteq \mathbb{P}$, we let

$$XZ(S) = \begin{cases} -1 & \text{if } S = \emptyset, \\ z_j & \text{if } S = \{j\}, \text{ and} \\ z_{j_1} \dots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1}j_i} - 1) & \text{if } S = \{j_1 < \dots < j_k\} \text{ where } k \geq 2. \end{cases} \quad (1)$$

We let

$$F_n^{\mathbb{P}}((\mathbf{x}, \mathbf{z})) := \sum_{w \in \mathbb{P}^n} z^w \prod_{i < j} x_{ji}^{\text{ji}(w)} \quad (2)$$

and

$$\mathcal{F}^{\mathbb{P}}((\mathbf{x}, \mathbf{z}), t) := 1 + \sum_{n \geq 1} F_n^{\mathbb{P}}((\mathbf{x}, \mathbf{z})) t^n = \sum_{w \in \mathbb{P}^*} t^{|w|} z^w \prod_{i < j} x_{ji}^{\text{ji}(w)}.$$

Then we can prove the following theorem.

Theorem 1.

$$\mathcal{F}^{\mathbb{P}}((\mathbf{x}, \mathbf{z}), t) = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} XZ(S)}. \quad (3)$$

In the case where $u = u_1 \dots u_j \in \mathbb{P}^*$, $u_1 > u_j$, and $\text{des}(u) = 1$, we can find closed expressions for the generating functions $N^{\mathbb{P}}(x, t)$, $N^k(x, t)$, $EN^{\mathbb{P}}(x, t)$ and $EN^k(x, t)$ by making appropriate substitutions for the variables z_i and x_{ij} in $\mathcal{F}^{\mathbb{P}}((\mathbf{x}, \mathbf{z}), t)$.

References

- [1] M. Jones and J.B. Remmel, A reciprocity method for computing generating function over the set of permutations with no consecutive occurrences of τ , Discrete Mathematics, **313** Issue 23 (2013), 2712-2729.

Patterns and Statistics

by

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027, USA
sagan@math.msu.edu

Let S_n , $n \geq 0$, be the symmetric groups or any other family of sets admitting a notion of pattern avoidance. Given $\Pi \subseteq S_n$, we let

$$S_n(\Pi) = \{\sigma \in S_n : \sigma \text{ avoids every element of } \Pi\}.$$

Also consider any statistic $\text{st} : S_n \rightarrow \{0, 1, 2, \dots\}$. Sagan and Savage suggested studying the generating functions

$$f_n^{\text{st}}(\Pi) = \sum_{\sigma \in S_n(\Pi)} q^{\text{st}(\sigma)}.$$

which are q -analogues of the numbers $|S_n(\Pi)|$. Of course, such cardinalities form a corner-stone of research in pattern avoidance.

For example, one can define Π and Π' to be *st-Wilf equivalent* if

$$f_n^{\text{st}}(\Pi) = f_n^{\text{st}}(\Pi')$$

for all $n \geq 0$. Note that st-Wilf equivalence implies ordinary Wilf equivalence. The first systematic study of st-Wilf equivalence was carried out by Dokos, Dwyer, Johnson, Sagan, and Selsor who completely determined the inv- and maj-Wilf equivalence classes for all Π consisting of permutations of length three. Interesting q -analogues of the Catalan numbers result when $|\Pi| = 1$, including ones which were studied previously by Carlitz and Riordan.

This talk will be a survey of results and conjectures concerning $f_n^{\text{st}}(\Pi)$.

Pattern Avoiding Set Partitions and Motzkin Left Factors

Mark Shattuck

Let $\mathcal{L}_n$ denote the set of all lattice paths of length n , using up, down and level steps, starting from the origin and not going below the x -axis. Such paths are often termed *Motzkin left factors*; see, e.g., [1, p. 111] or [3, p. 9]. Let $L_n = |\mathcal{L}_n|$ if $n \geq 1$, with $L_0 = 1$. The numbers L_n are also given equivalently by the generating function

$$\sum_{n \geq 0} L_n x^n = \frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)}. \quad (1)$$

The L_n enumerate a variety of other structures, ranging from the set of directed animals of size n to the permutations of $[n] = \{1, 2, \dots, n\}$ simultaneously avoiding 321 and the barred pattern $4\bar{1}523$ to the set of base 3 n -digit numbers whose digit sum is also n . See A005773 of OEIS for further information on these numbers. Here, we provide new combinatorial interpretations for the L_n sequence in terms of finite set partitions, showing, in particular, that it enumerates certain two-pattern avoidance classes.

Let P_n denote the set of all partitions of $[n]$. A member of P_n is said to *avoid* a pattern τ if its canonical representation as a restricted growth function contains no subsequences isomorphic to τ . If $\{w_1, w_2, \dots\}$ is a set of classical patterns, then let $P_n(w_1, w_2, \dots)$ denote the subset of P_n that avoids all of the patterns and let $p_n(w_1, w_2, \dots) = |P_n(w_1, w_2, \dots)|$. We have the following avoidance result.

Theorem 0.1. *If $n \geq 0$, then $p_n(u, v) = L_n$ for the following sets (u, v) :*

- (i) (1222, 1212) (ii) (1112, 1212)
- (iii) (1211, 1221) (iv) (1222, 1221).

We remark that in the first two cases, our proofs entail defining explicit bijections between the subclass of partitions in question and the set of Motzkin left factors. In the latter two cases, the proofs are more algebraic and involve applications of the *kernel method* [2] to solve the functional equations that arise once a certain parameter has been introduced. Furthermore, we are able to prove a refinement of Theorem 0.1 by adding a variable marking the number of blocks. In addition, one can obtain p, q -generalizations of L_n and the Motzkin numbers M_n by considering pairs of statistics on the sets $P_n(1222, 1212)$ and $P_n(111, 1212)$, respectively.

Finally, by numerical evidence, there are no other members of the $(4, 4)$ -Wilf equivalence class for set partitions corresponding to the numbers L_n . We have also considered the comparable question for other Wilf classes and have been able to identify all of the $(4, 4)$ -Wilf equivalence classes for set partitions, with enumerations in several cases. (This is joint work with Toufik Mansour.)

References

- [1] L. Alonso and R. Schott, *Random Generation of Trees*, Kluwer Academic Publishers, Dordrecht, The Netherlands 1995.
- [2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps, Generating functions for generating trees, (*Formal Power Series and Algebraic Combinatorics*, Barcelona, 1999), *Discrete Math.* **246:1-3** (2002) 29–55.
- [3] M. Josuat-Vergès and M. Rubey, Crossings, Motzkin paths, and moments, arXiv:1008.3093v1 [math.CO].

Intervals of Permutations with a Fixed Number of Descents are Shellable

Jason P Smith

University of Strathclyde, Glasgow, UK

jason.p.smith@strath.ac.uk

Abstract

The set of all permutations, ordered by pattern containment, is a poset. We introduce an order isomorphism from the poset of permutations with a fixed number of descents to a certain poset of words with subword order. We use this bijection to show that intervals of permutations with a fixed number of descents are shellable and present a formula for the Möbius function of these intervals.

1 Introduction

We present some results from [Smi14a] which is an extension of the work in [Smi14b]. We consider the poset of permutations $\mathcal{P}$ with the partial ordering $\sigma \leq \pi$ if σ occurs as a pattern in π . An *interval* $[\sigma, \pi]$ in $\mathcal{P}$ is a subposet consisting of all permutations $z \in \mathcal{P}$ with $\sigma \leq z \leq \pi$. A *chain* in a poset P is a totally ordered subset $C = \{c_1 < \dots < c_t\}$. For example, $21 < 2341 < 24513$ is a chain in $[1, 24513]$. The *length* of a chain C is given as $\text{len}(C) = |C| - 1$ and the *rank* of a poset is the length of the longest chain in that poset. A *descent* occurs at i in a permutation $\pi = \pi_1 \dots \pi_n$ if $\pi_i > \pi_{i+1}$. As an example, 23154 has descents at 2 and 4. It is straightforward to show that if the permutations σ and π both have exactly k descents then any permutation $\tau \in [\sigma, \pi]$ also has exactly k descents. We denote the induced subposet of all permutations with exactly k descents as $\mathcal{P}_k$. The Möbius function for a poset is defined as: $\mu(a, b) = 0$ if $a \not\leq b$, $\mu(a, a) = 1$ for all a

and for $a < b$:

$$\mu(a, b) = - \sum_{a \leq z < b} \mu(a, z).$$

Given an interval $\mathcal{I} = [\sigma, \pi]$ we construct a simplicial complex $\Delta(\mathcal{I})$, called the *order complex* of $\mathcal{I}$, whose vertices are the elements of the interior of $\mathcal{I}$, that is, elements of $(\sigma, \pi) := [\sigma, \pi] \setminus \{\sigma, \pi\}$, and whose faces are the chains of (σ, π) . When we attribute a topological property to an interval we mean the corresponding property of its order complex. We refer the reader to [Wac07] for extensive background on the subject of order complexes.

A simplicial complex is *pure* if all its maximal faces, which are called *facets*, have the same dimension. The order complex of an interval of permutations is always pure. A pure simplicial complex Δ is *shellable* if its facets can be arranged in linear order $F_1, F_2, \dots, F_t$ in such a way that the subcomplex $(\cup_{i=1}^{k-1} \langle F_i \rangle) \cap \langle F_k \rangle$ is pure and $(\dim \Delta - 1)$ -dimensional for all $k = 2, \dots, t$, where $\langle F \rangle = \{G : G \subseteq F\}$, that is, $\langle F \rangle$ is the subcomplex generated by F . Again we refer the reader to [Wac07] for extensive background on the subject of shellability.

Let $\mathcal{A}$ be the poset of words on the alphabet of positive integers, with the partial order called *subword order* where if two words v and $w = w_1 \dots w_n$ belong to $\mathcal{A}$ then $v \leq w$ if there is a subsequence $w_{i_1} \dots w_{i_m}$ in w such that $v = w_{i_1} \dots w_{i_m}$. For example, $2132 \leq 212312$ but $2132 \not\leq 21233$. In [Bjö88] a formula is given for computing the Möbius function on intervals of $\mathcal{A}$ in polynomial time and it is shown that all intervals in $\mathcal{A}$ are shellable. We present an order isomorphism, that is, an order preserving bijection, between each interval in the permutation posets $\mathcal{P}_k$ and a corresponding interval in $\mathcal{A}$. This allows us to easily compute the Möbius function of intervals from the posets $\mathcal{P}_k$ and to show they are shellable.

The *Betti number* $\beta_k(X)$ of a simplicial complex X is the rank of the k -th homology group of X (for background on the homology of simplicial complexes we refer the reader to [Koz08]). The Philip Hall Theorem and the Euler-Poincaré formula, which appear as Proposition 1.2.6 and Theorem 1.2.8 in [Wac07], combined state:

$$\mu(I) = \tilde{\chi}(\Delta(I)) = \sum_{i=-1}^{\dim \Delta(I)} (-1)^i \beta_i(\Delta(I)), \quad (1)$$

where I is an interval and $\tilde{\chi}(\Delta(I))$ is the reduced Euler characteristic of the order complex of I .

There is a *generalised subword order*, defined in [SV06], where we take a poset P and let P^* denote the poset of finite words whose letters are elements of P . If $u, w \in P^*$ then $u \leq_{P^*} w$ if there is a subword $w_{i_1} \dots w_{i_\ell}$ having length $\ell = |u|$ such that $u_j \leq_P w_{i_j}$, for all $1 \leq j \leq \ell$. If P is an antichain then generalised subword order is equal to subword order. In [SV06] a formula is presented for the Möbius function of words with generalised subword order when P is a chain. It is also shown there is an order isomorphism between intervals of these words and intervals of *layered permutations*, that is permutations that can be expressed as a direct sum of decreasing permutations, such as $1 \oplus 21 \oplus 321 \oplus 1 \oplus 1 = 13265478$.

In [Bjö80] it is shown that if an interval $\mathcal{I}$, in any poset, contains a *non-trivial* disconnected subinterval, that is, a disconnected subinterval of rank at least three, then $\mathcal{I}$ is not shellable. The first major result on the topology of intervals from the poset $\mathcal{P}$ is given in [MS13], where it is shown that if P is a rooted forest then any interval $[u, v]$ in P^* that does not contain a non-trivial disconnected subinterval is shellable. This result is then used to show that intervals of layered permutations that do not contain a non-trivial disconnected subinterval are shellable and it is conjecture that the same applies to the more general class of separable permutations.

We consider the following subposet of words with subword order, by the results from [Bjö88] we know that intervals of this poset are shellable and have an effective formula for their Möbius function.

Definition 1. Let $\widehat{\mathcal{A}}$ denote the poset of words with subword order on the alphabet of all positive integers with the additional conditions that for any $w \in \widehat{\mathcal{A}}$:

- There is at least one occurrence of each letter $i \in \{1, \dots, \max(w)\}$.
- The rightmost occurrence of each letter $i \in \{1, \dots, \max(w) - 1\}$ is preceded by an occurrence of $i + 1$.

Let $\widehat{\mathcal{A}}_k$ denote the subposet of $\widehat{\mathcal{A}}$ of words w where $\max(w) = k$.

We present an order isomorphism between intervals of $\mathcal{P}_k$ and intervals of $\widehat{\mathcal{A}}_{k+1}$, defined as:

$$f : \mathcal{P} \rightarrow \widehat{\mathcal{A}} : \pi \mapsto d_\pi(1)d_\pi(2) \dots d_\pi(|\pi|),$$

$$f^{-1} : \widehat{\mathcal{A}} \rightarrow \mathcal{P} : w \mapsto [i : w_i = 1][i : w_i = 2] \dots [i : w_i = \max(w)].$$

Where $d_\pi(c)$ is one plus the number of descents preceding the letter c in the permutations π and $[i : w_i = t][i : w_i = t + 1]$ is the element of the set $\{i : w_i = t\}$ in increasing order followed by the elements of the set $\{i : w_i = t + 1\}$ in increasing order. For example, if $\pi = 263415$ then $f(\pi) = 312231$ and if $w = 214321$ then $g(w) = 261543$.

We can use this order isomorphism along with [Bjö88, Theorem 3] to get:

Corollary 1. *Any interval $[\sigma, \pi]$, where σ and π are permutations with the same number of descents, is dual CL-shellable.*

Using [Bjö88, Theorem 1] and our order isomorphism we can compute the Möbius function of these intervals. To do so we need the following definition:

Definition 2. *An adjacency in a permutation is a sequence of consecutively valued letters in increasing consecutive order. The tail of an adjacency is all but the first letter of the adjacency. An occurrence of pattern in π is normal if the occurrence contains all the tails of the adjacencies in π . We denote the number of normal occurrences of σ in π as $\binom{\pi}{\sigma}_n$.*

The adjacencies in the permutation 142356 are 23 and 56 so the tails of the adjacencies are 3 and 6. The only normal occurrence of 213 in 142356 is 436, therefore $\binom{142356}{213}_n = 1$.

Proposition 1. *If σ and π are permutations with the same number of descents, then*

$$\mu(\sigma, \pi) = (-1)^{|\pi| - |\sigma|} \binom{\pi}{\sigma}_n.$$

References

- [Bjö80] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Transactions of the American Mathematical Society*, 260(1):159–183, 1980.
- [Bjö88] Anders Björner. The Möbius function of subword order. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, volume 19 of *IMA Vol. Math. Appl.*, pages 118–124. Springer, New York, 1988.
- [Koz08] D. Kozlov. *Combinatorial Algebraic Topology*. Algorithms and Computation in Mathematics. Springer, 2008.

- [MS13] Peter R. W. McNamara and Einar Steingrímsson. On the topology of the permutation pattern poset. *arXiv*, 2013. arXiv:1305.5569 [math.CO].
- [Smi14a] Jason P. Smith. Intervals of permutations with a fixed number of descents are shellable. 2014. arXiv:1405.2560v3 [math.CO].
- [Smi14b] Jason P. Smith. On the Möbius function of permutations with one descent. *The Electronic Journal of Combinatorics*, 21:2.11, 2014.
- [SV06] Bruce E. Sagan and Vincent Vatter. The Möbius function of a composition poset. *J. Algebraic Combin.*, 24(2):117–136, 2006.
- [Wac07] Michelle L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.

An involution on involutions

Miklós Bóna and Rebecca Smith*

We consider the involution on involutions obtained by first applying the Robinson-Schensted-Knuth Algorithm to a given involution π to obtain the associated Standard Young Tableau $P(\pi)$, then taking the transpose of $P(\pi)$, and finally applying the inverse of the RSK Algorithm to $P(\pi)^T$. We will be very original in our naming of this bijection and call it f .

As we are dealing exclusively with involutions, we will classify each entry as a fixed point, a small entry, or a large entry. Fixed points will be defined as per usual. All other entries of a permutation will be in 2-cycles with the smaller entry of any given 2-cycle being a small entry and the larger entry being a large entry.

In the case when the the reverse of π is also an involution, Schensted [1] showed that $f(\pi) = \pi^r$. We look to define this bijection in general without relying on the RSK algorithm. To this end, we look at some other special cases to learn more.

Some basic relationships between an involution and its corresponding insertion tableau are given below:

Proposition 1 *Suppose the insertion tableau $P(\pi)$ is such that n is alone in the last column. Then every maximum length increasing subsequence of π ends with n .*

Corollary 2 *Suppose the insertion tableau $P(\pi)$ is such that n is alone in the last row. Then every maximum length decreasing subsequence of π begins with n .*

Note that the converses of Proposition 1 and Corollary 2 are also true. Another entry in the last column of the $P(\pi)$ (respectively the last row of $P(\pi)$) regardless of whether or not n is also there would indicate that the entry n was not necessary to complete an increasing (respectively, decreasing) subsequence of maximum length.

Corollary 3 *The number of involutions with fixed point n is the same as the number of involutions with all maximum length decreasing subsequences beginning with n .*

We are also able to define $f(\pi)$ in the special case when π is an involution that has both its smallest and largest elements as fixed points.

Proposition 4 *If we have that both 1 and n are fixed points of an involution, we can apply the map to the remaining portion of the involution knowing that 1 and n will be in a 2-cycle. Basically, $f(1 \oplus \pi \oplus n) = (1n)f(\pi)$ where the entries of $f(\pi)$ are shifted up by 1.*

There are other results one can obtain with similar methods when very specific elements of the involution are fixed, but this does not appear to extend as completely as would be desired.

Moving on, we find a simple relationship between descents of π and ascents of $f(\pi)$. The same method also gives the corresponding relationship between the positions of consecutive values of elements.

Proposition 5 *If π has a descent at i , then $f(\pi)$ has an ascent at i .*

Proposition 6 *If i and $i+1$ are in ascending order as a subsequence in π , then i and $i+1$ will appear in descending order as a subsequence in $f(\pi)$.*

However, this information is not enough to completely define our bijection in general. We move on to the case of 321-avoiding and 123-avoiding involutions.

If π avoids 321, then we know its Standard Young Tableau must have at most two rows. Thus the Standard Young Tableau of $f(\pi)$ will have at most two columns and hence avoid 123.

Proposition 7 *The first row of the Standard Young Tableau of a 321-avoiding involution π consists all small entries and all fixed points. The second row therefore consists of all large entries.*

Proposition 8 *Similarly, let $\tau = f(\pi)$ be a 123-avoiding involution. To get τ back from its tableau, if n is in the second column (not a fixed point) we match it with the top entry of the second column (smallest large entry) to form a two cycle.*

With some more effort (and definitions), we can show how define f when it is applied to 123-avoiding permutations without the use of the RSK algorithm. However, at the current time, a definition of the inverse of f (without the use of RSK) in this case remains a mystery.

References

- [1] SCHENSTED, C. Longest increasing and decreasing subsequences. *Canad. J. Math.* 13 (1961), 179–191.

q-Analogues of Convolutions of Fibonacci Numbers
 Janine LoBue Tiefenbruck
 University of California, San Diego

We study the simplest kind of frame pattern, called the μ pattern, in words $w = w_1w_2 \dots w_n$ over the alphabet $[k] = \{1, 2, \dots, k\}$. Given a word $w = w_1w_2 \dots w_n \in [k]^*$, say that the pair (w_i, w_j) is an occurrence of the pattern μ in w if $i < j$, $w_i < w_j$, and there is no $i < \ell < j$ such that $w_i \leq w_\ell \leq w_j$. We say a μ -match in w is a *trivial* μ -match if $w_j = w_i + 1$ and it is a *nontrivial* μ -match if $w_j > w_i + 1$.

Let $NR([k])$ denote the set of words over the alphabet $[k]$ with no consecutive repeated letters. We are particularly interested in counting occurrences of μ in words in $NR([k])$ because one can reduce the problem of counting trivial or non-trivial μ -matches in arbitrary words to counting those matches in words with no consecutive repeats. For example, the word 2122233 has the same number of trivial and nontrivial μ -matches as the word 2123.

For each word w in $NR([k])$, let the weight of w be given by $t^{|w|}p^{ntriv(w)}q^{triv(w)}$, where $|w|$ is the length of w , $ntriv(w)$ is the number of nontrivial μ -matches in w and $triv(w)$ is the number of trivial μ -matches in w . We study the generating functions $N^{i,j}(p, q, t)$ that sum the weights of all words w in $NR([3])$ that start with the letter i and end in the letter j . In particular, we show that $N^{3,1}(p, 1, t) = \sum_{r \geq 0} \frac{t^{2r+2}p^r}{(1-t-t^2)^{r+1}}$ so that the number of words in $NR([3])$ starting with 3, ending with 1, and having r nontrivial μ -matches is counted by the convolutions of $r + 1$ copies of the Fibonacci numbers. It follows that the coefficient of p^r in $N^{3,1}(p, q, t)$ is a q -analogue of the generating function of the convolution of $r + 1$ copies of the Fibonacci numbers.

In this talk, I will indicate how to compute the generating functions $N^{i,j}(p, q, t)$ for all $i, j \in [3]$ and show that many well-known sequences appear as their coefficients. We can also extend our results using similar methods to obtain the analogous generating functions in the case $k = 4$, and doing so yields a q -analogue of the generating function of the convolution of odd-indexed Fibonacci numbers.

This is joint work with Jeffrey Remmel.

On poset structures of set partitions which avoid a pattern of length 3

Masaya Tomie

Morioka University, Takizawa-shi, Iwate 020-0183, Japan
(e-mail: tomie@morioka-u.ac.jp)

Abstract

In this extended abstract, we focus on the poset structure of set partitions which avoid a pattern of length 3 with the refinement order. Especially our main interest is the poset of the 112-avoiding set partitions. We provide an inductive construction of the Hasse diagram of the 112-avoiding set partitions of $[n]$ and we discuss several properties of the poset, the number of maximal chains, self-duality, and the Möbius numbers. We also give two Gray codes for 112-avoiding set partitions of given size. Finally, we define an involution on $\mathcal{P}_n(112)$ and as an application, we obtain equations for some statistic.

1 Introduction

In 1968, Knuth gave the number of n -permutations avoiding a pattern of length 3, which is the n -th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ [6]. The first systematic study of pattern avoidance was done by Simion and Schmidt in [9], where they found the number of n -permutations that avoid any subset of patterns of length 3. The subject permutation patterns is useful language in enumerative combinatorics, algebraic combinatorics, algebraic geometry and computer science, see Kitaev's text [4].

In 1996, Klazar studied the subsequence pattern avoidance on set partitions, where he determined the number of partitions of $[n]$ that avoid the subsequence pattern 1212 or that of 1221 and he showed that the number in both cases is given by the n -th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ [5]. In 2007, Sagan determined the number of partitions of $[n]$ which avoid the subsequence pattern 111, 112, 121, 122, and 123 in [8]. Goyt gave the number of set partitions of $[n]$ that avoids a subset of patterns of length 3 in [2]. There are many results about the enumeration of pattern avoidance on set partitions,

see [7] and the references therein. We denote the set of τ -avoiding set partitions of $[n]$ by $\mathcal{P}_n(\tau)$.

Definition 1. Let $\widehat{\mathcal{P}}_n(\tau)$ denote the poset of τ -avoiding set partitions of $[n]$ with the refinement order.

In this extended abstract, we consider the poset structures of $\widehat{\mathcal{P}}_n(\tau)$, where τ is a restricted growth function of length 3, i.e., $\tau \in \{111, 112, 122, 121, 123\}$. For $\tau \in \{111, 121, 122, 123\}$, the poset structures of $\widehat{\mathcal{P}}_n(\tau)$ are quite simple and therefore we focus on the set partitions that avoid the pattern 112.

1.1 The case of $\widehat{\mathcal{P}}_n(\tau)$ with $\tau \in \{111, 121, 122, 123\}$

The poset structures of $\widehat{\mathcal{P}}_n(\tau)$ with $\tau \in \{111, 121, 122, 123\}$ are quite simple.

Proposition 1. For $\tau = 111, 121, 122$, and 123, the poset $\widehat{\mathcal{P}}_n(\tau)$ is a graded and bounded poset.

1. The rank of $\widehat{\mathcal{P}}_n(123)$ is equal to 2 with the minimum (resp. maximum) element $1/2/\cdots n$ (resp. $12\cdots n$).
2. $\widehat{\mathcal{P}}_n(121)$ and $\widehat{\mathcal{P}}_n(122)$ are isomorphic to the Boolean algebra of rank $n - 1$.
3. $\widehat{\mathcal{P}}_n(111)$ is the poset of matchings in $[n]$ with the inclusion order. Let μ_n denote the Möbius number of $\widehat{\mathcal{P}}_n(111) \cup \{\widehat{1}\}$, where $\widehat{1}$ is a unique maximum element, then we have $\sum_{n \geq 0} \frac{\mu(n)}{n!} x^n = -\exp(x - \frac{x^2}{2})$.

2 The poset structures of $\widehat{\mathcal{P}}_n(112)$

Sagan showed that the number of $\mathcal{P}_n(112)$ is 2^{n-1} in [8]. We focus on the poset structure of $\widehat{\mathcal{P}}_n(112)$. Remark that $\widehat{\mathcal{P}}_n(112)$ is a graded and bounded poset of rank $n - 1$ with the minimum element $1/2/\cdots/n$ and the maximum element $12\cdots n$. We introduce an inductive construction for $\widehat{\mathcal{P}}_n(112)$.

Define

$$\widehat{\mathcal{P}}_n^{(1)}(112) := \{B_1/B_2/\cdots/B_k \vdash [n] \mid B_1 = \{1\}\} \quad (1)$$

$$\widehat{\mathcal{P}}_n^{(2)}(112) := \{B_1/B_2/\cdots/B_k \vdash [n] \mid |B_1| \geq 2\} \quad (2)$$

$$\begin{aligned}\psi_{n-1}^{(1)} : \widehat{\mathcal{P}}_{n-1}(112) &\rightarrow \widehat{\mathcal{P}}_n^{(1)}(112) \\ \psi_{n-1}^{(1)}(B_1/B_2/\cdots/B_k) &= 1/B_1^{+1}/B_2^{+1}/\cdots/B_k^{+1}\end{aligned}\quad (3)$$

$$\begin{aligned}\psi_{n-1}^{(2)} : \widehat{\mathcal{P}}_{n-1}(112) &\rightarrow \widehat{\mathcal{P}}_n^{(2)}(112) \\ \psi_{n-1}^{(2)}(B_1/B_2/\cdots/B_k) &= B_1 \cup \{n\}/B_2/\cdots/B_k,\end{aligned}\quad (4)$$

where $B^{+1} := \{i+1 | i \in B\}$. Remark that $\widehat{\mathcal{P}}_n(112)$ is a disjoint union of $\widehat{\mathcal{P}}_n^{(1)}(112)$ and $\widehat{\mathcal{P}}_n^{(2)}(112)$ as a set.

Lemma 1. *The map $\psi_{n-1}^{(1)}$ (resp. $\psi_{n-1}^{(2)}$) is a poset isomorphism between $\widehat{\mathcal{P}}_{n-1}(112)$ and $\widehat{\mathcal{P}}_n^{(1)}(112)$ (resp. $\widehat{\mathcal{P}}_n^{(2)}(112)$) when regarded as a subposet of $\widehat{\mathcal{P}}_n(112)$.*

Lemma 2. *For $\sigma, \tau \in \widehat{\mathcal{P}}_n(112)$, suppose that τ covers σ , we write $\sigma \prec \tau$, then we have $\tau, \sigma \in \widehat{\mathcal{P}}_n^{(i)}(112)$ and $(\psi_{n-1}^{(i)})^{-1}(\sigma) \prec (\psi_{n-1}^{(i)})^{-1}(\tau)$ in $\widehat{\mathcal{P}}_n(112)$, where $i \in \{1, 2\}$ or $\sigma = 1/2/\cdots/i, (i+1), \dots, n \in \mathcal{P}_n^{(1)}(112)$ and $\tau = 1, i, (i+1), \dots, n/2/\cdots/i-1 \in \widehat{\mathcal{P}}_n^{(2)}(112)$ for some $2 \leq i \leq n$.*

By Lemma 1 and Lemma 2, we can construct $\widehat{\mathcal{P}}_n(112)$ from $\widehat{\mathcal{P}}_{n-1}(112)$.

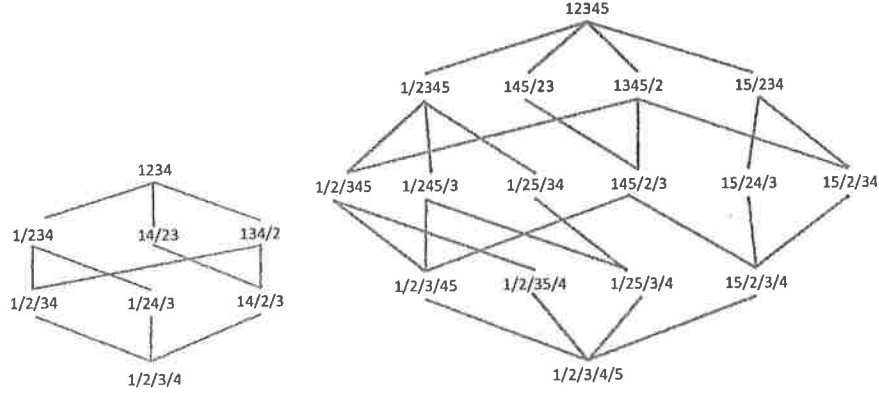


Figure 1: Hasse diagrams of $\mathcal{P}_4(112)$ and $\mathcal{P}_5(112)$

Remark 1. *Simion and Schmidt defined the $S_n(132)$ -chain-permutational lattice $L_n(132)$, whose maximal chains are in bijection with the 132-avoiding permutations of length n [9]. Our construction of $\widehat{\mathcal{P}}_n(112)$ is similar to that of $L_{n-1}(132)$. For*

$n \geq 5$, there is an interval $[x, \hat{1}] \subset L_{n-1}(132)$, where $\hat{1}$ is the maximum element and x is an atom, such that $[x, \hat{1}]$ is isomorphic to the chain of length $n - 2$ and also for every atom $y \in \hat{\mathcal{P}}_n(112)$, the interval $[y, 12 \cdots n]$ is not poset isomorphic to the chain. Therefore $\hat{\mathcal{P}}_n(112)$ and $L_{n-1}(132)$ are not isomorphic for $n \geq 5$.

By the construction, we can compute the number of maximal chains.

Proposition 2. *The number of maximal chains of $\hat{\mathcal{P}}_n(112)$ is equal to $\frac{1}{n-1} \binom{2n-2}{n-1}$, i.e., the $(n - 1)$ -th Catalan number.*

This object seems to be a new Catalan object.

Theorem 1. *$\hat{\mathcal{P}}_n(112)$ is a self dual lattice.*

We can obtain the Möbius number of $\hat{\mathcal{P}}_n(112)$ via the theory of NBB bases [1].

Theorem 2. *Let μ_n denote the Möbius number of $\hat{\mathcal{P}}_n(112)$. Then we have $\mu_1 = 1, \mu_2 = -1, \mu_3 = 1$, and $\mu_n = 0$ for $n \geq 4$.*

3 Gray codes for $\mathcal{P}_n(112)$

In this section, we provide two Gray codes for $\mathcal{P}_n(112)$.

We consider the Hasse diagram of $\hat{\mathcal{P}}_n(112)$ as a graph, the vertices are partitions in $\mathcal{P}_n(112)$ and the edges are the cover relations, i.e., π and τ are adjacent if and only if $\pi \prec \tau$ or $\tau \prec \pi$.

For $\sigma, \tau \in \hat{\mathcal{P}}_n(112)$, define $d(\sigma, \tau)$ to be the length of a shortest path from σ to τ in the Hasse diagram of $\hat{\mathcal{P}}_n(112)$.

Theorem 3. *We have two Gray codes of distance 2 for $\hat{\mathcal{P}}_n(112)$, one starts from $1/2/\cdots/n$ and ends at $1, n/2/3/\cdots/(n-1)$ and the other starts from $1/2/\cdots/n$ and ends at $12 \cdots n$.*

We define a graph $G_n(112)$ whose vertices are the partitions in $\mathcal{P}_n(112)$ and whose edges are the pairs such that they differ by taking an element from one block and moving it into another block, or creating a new singleton. This adjacent relation is more restrictive than that for noncrossing partitions defined by Huemer et al.(2009), where they constructed a Gray code for noncrossing partitions [3].

Theorem 4. *The graph $G_n(112)$ has two Hamilton paths. In other words, $\mathcal{P}_n(112)$ has two Gray codes such that successive partitions differ by taking an element from one block and moving it into another block, or creating a new singleton.*

Example 1. For $n = 4$ we have the following Gray codes of distance 2.

- (1) $1/2/3/4 \rightarrow 1/2/3, 4 \rightarrow 1/2, 4/3 \rightarrow 1/2, 3, 4 \rightarrow 1234 \rightarrow 1, 3, 4/2 \rightarrow 1, 4/2, 3 \rightarrow 1, 4/2/3,$
 (2) $1/2/3/4 \rightarrow 1/2/3, 4 \rightarrow 1/2, 3, 4 \rightarrow 1/2, 4/3 \rightarrow 1, 4/2/3 \rightarrow 1, 4/2, 3 \rightarrow 1, 3, 4/2 \rightarrow 1234.$

For $n = 5$ we have the following Gray codes of distance 2.

- (3) $1/2/3/4/5 \rightarrow 1/2/3/4, 5 \rightarrow 1/2/3, 5/4 \rightarrow 1/2/3, 4, 5 \rightarrow 1/2, 3, 4, 5 \rightarrow 1/2, 4, 5/3 \rightarrow 1/2, 5/3, 4 \rightarrow 1/2, 5/3/4 \rightarrow 1, 5/2/3/4 \rightarrow 1, 5/2/3, 4 \rightarrow 1, 5/2, 3, 4 \rightarrow 1, 5/2, 4/3 \rightarrow 1, 4, 5/2/3 \rightarrow 1, 4, 5/2, 3 \rightarrow 1, 3, 4, 5/2 \rightarrow 12345,$
 (4) $1/2/3/4/5 \rightarrow 1/2/3/4, 5 \rightarrow 1/2/3, 4, 5 \rightarrow 1/2/3, 5/4 \rightarrow 1/2, 5/3/4 \rightarrow 1/2, 5/3, 4 \rightarrow 1/2, 4, 5/3 \rightarrow 1/2, 3, 4, 5 \rightarrow 12345 \rightarrow 1, 3, 4, 5/2 \rightarrow 1, 4, 5/2, 3 \rightarrow 1, 4, 5/2/3 \rightarrow 1, 5/2, 4/3 \rightarrow 1, 5/2, 3, 4 \rightarrow 1, 5/2/3, 4 \rightarrow 1, 5/2/3/4.$

4 Appendix An involution on $\mathcal{P}_n(112)$

In this section, we introduce an involution on $\mathcal{P}_n(112)$. First, we consider a set of tuples of pairs of non negative integers as follows:

$$T_n(112) := \{((1, a_1), (2, a_2), \dots, (k, a_k)) \mid a_i \geq 0, \sum_{1 \leq i \leq k} (a_i + 1) = n\}. \quad (5)$$

For $(B_1, B_2, \dots, B_k) \in \mathcal{P}_n(112)$ with $\min B_1 < \min B_2 < \dots < \min B_k$, the k -tuples $((\min B_1, |B_1| - 1), (\min B_2, |B_2| - 1), \dots, (\min B_k, |B_k| - 1))$ is an element of $T_n(112)$ and this correspondence is a bijection between $\mathcal{P}_n(112)$ and $T_n(112)$. Next we define a map Inv_{T_n} on $T_n(112)$.

For $((1, a_1), (2, a_2), \dots, (k, a_k)) \in T_n(112)$ with $1 \leq a_{i_1}, a_{i_2}, \dots, a_{i_x}$ and $i_1 < i_2 < \dots < i_x$, define $\text{Inv}_{T_n}(((1, a_1), (2, a_2), \dots, (k, a_k))) := ((1, a'_1), (2, a'_2), \dots, (k, a'_k))$, where $a'_{i_t} = a_{i_{x+1-t}}$ for $1 \leq t \leq x$ and $a'_j = a_j$, if $j \notin \{i_1, i_2, \dots, i_x\}$; indeed, $a_j = a'_j = 0$. It is easy to see that our Inv_{T_n} is an involution. We can define an involution $\text{Inv}_{\mathcal{P}_n}$ on $\mathcal{P}_n(112)$ via the above bijection between $T_n(112)$ and $\mathcal{P}_n(112)$ and the involution on $T_n(112)$.

For a restricted growth function $\pi = \pi_1 \pi_2 \dots \pi_n$ and a sequence $\sigma = \sigma_1 \sigma_2 \dots \sigma_k$, a subword $\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}$ is said to be a *non trivial occurrence* of σ if $st(\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}) = st(\sigma_1 \sigma_2 \dots \sigma_k)$ and each π_{i_j} is not a smallest element in some block of the partition corresponding to π , see Definition 4.20 in [7].

Definition 2. For $\pi \in \mathcal{P}_n(112)$, whose canonical sequential form is $\pi_1\pi_2 \cdots \pi_n$, and a sequence $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$, define $\text{nocc}_\pi(\sigma)$ to be the number of non trivial occurrence of σ in $\pi_1\pi_2 \cdots \pi_n$.

For instance, if $\pi = 1, 6, 7, 8/2, 4, 5/3 \vdash [8]$, then the corresponding canonical sequential form $\pi_1\pi_2\pi_3\pi_4\pi_5\pi_6\pi_7\pi_8$ is 12322111 and the non trivial occurrences of 11 are $\pi_4\pi_5$, $\pi_6\pi_7$, $\pi_6\pi_8$, and $\pi_7\pi_8$, hence $\text{nocc}_{1,6,7,8/2,4,5/3}(11) = 4$. Our bijection provides the following non trivial equation.

Theorem 5. For a sequence $a_1a_2 \cdots a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k$, we have

$$\sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(a_1a_2 \cdots a_k)} = \sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(a'_1a'_2 \cdots a'_k)}, \quad (6)$$

where $a'_i = a_1 + 1 - a_{k+1-i}$ for $1 \leq i \leq k$.

Remark that if a sequence $a_1a_2 \cdots a_k$ does not satisfy $a_1 \geq a_2 \geq \cdots \geq a_k$, then $\text{nocc}_\pi(a_1a_2 \cdots a_k) = 0$ for $\pi \in \mathcal{P}_n(112)$.

Corollary 1. We have

$$\sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(211)} = \sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(221)}, \quad (7)$$

$$\sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(2111)} = \sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(2221)}, \quad (8)$$

$$\sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(3211)} = \sum_{\pi \in \mathcal{P}_n(112)} q^{\text{nocc}_\pi(3321)}. \quad (9)$$

References

- [1] A. Blass, B. Sagan, Möbius functions of lattices. *Adv. Math.* 127, 94-123 (1997).
- [2] A. Goyt, Avoidance of partitions of a three-element set, *Advances in Applied Mathematics* 41 (2008) 95-114 .
- [3] G. Huemer, F. Hurtado, M. Noy and E. Omaña-Pulido, Gray codes for non-crossing partitions and dissections of a convex polygon, *Discrete Applied Mathematics* 157(7), (2009), 1509-1520.
- [4] S. Kitaev, Patterns in permutations and words, Springer-Verlag, 2011.

- [5] M. Klazar, On abab-free and abba-free set partitions, *European. J. Combin.* 17 (1996) 53-68.
- [6] D. Knuth, The art of computer programming, vol 1, *Addison-Wesley, Reading 1973, Addison-Wesley Series in Computer Science and Information Processing.*
- [7] T. Mansour, Combinatorics of Set Partitions, *Discrete Mathematics and its Applications, Chapman Hall/CRC, Taylor Francis Group, Boca Raton, London, New York*
- [8] B. E. Sagan, Pattern Avoidance in Set Partitions, *Ars Combinatoria*, 94, 79-06, 2010.
- [9] R. Simion, F. Schmidt, Restricted permutations, *European journal of Combinatorics*, 6, 383-406, 1985.
- [10] R. P. Stanley, Enumerative Combinatorics, vol.1, *Cambridge University Press, Cambridge, 1997.*

A HOPF ALGEBRA OF PERMUTATION PATTERN FUNCTIONS (EXTENDED ABSTRACT)

YANNIC VARGAS

LABORATOIRE DE COMBINATOIRE ET D'INFORMATIQUE MATHÉMATIQUE (LACIM)
UNIVERSITÉ DU QUÉBEC À MONTRÉAL (UQAM)

Let $\mathbb{N}$ the set of nonnegative integers. For $n \in \mathbb{N}$, consider $\mathfrak{S}_n$ the set of permutations on $\{1, 2, \dots, n\}$. We denote λ the empty permutation, thus $\mathfrak{S}_0 = \{\lambda\}$. Also, let $\mathfrak{S} = \cup_{n \geq 0} \mathfrak{S}_n$ (disjoint union) the set of all permutations. Let $\mathbb{Q}^{\mathfrak{S}}$ be the set of all functions from $\mathfrak{S}$ to $\mathbb{Q}$. This is a commutative $\mathbb{Q}$ -algebra when it is equipped by sum and the pointwise product: if $f, g \in \mathbb{Q}^{\mathfrak{S}}$ and $\sigma \in \mathfrak{S}$, then

$$(f + g)(\sigma) = f(\sigma) + g(\sigma) \quad , \quad (f \cdot g)(\sigma) := f(\sigma)g(\sigma).$$

In this work we consider a special subalgebra of $\mathbb{Q}^{\mathfrak{S}}$ based on permutation patterns. If $w, \sigma \in \mathfrak{S}$, then σ is said to be a **pattern** of w if there exists a subsequence of entries of w that has the same relative order as σ . We define $\{w_\sigma\}$ as the number of occurrences of σ as a pattern of w . For example,

$$\left\{ \begin{smallmatrix} 25143 \\ 132 \end{smallmatrix} \right\} = |\{254, 253, 143\}| = 3.$$

Notice that $\{w_\lambda\} = 1$ and $\{w_\sigma\} = 1$, for all $w \in \mathfrak{S}$. Permutation patterns are considered in several domains, such as as theoretical computer science and enumerative combinatorics; namely, in problems like stack-sorting (Knuth, 1968 [8]) and pattern avoidance. This work is intended to study permutation patterns from an algebraic combinatorics point of view. The main result of this article is the following:

Theorem. *The family of functions $w \mapsto \{w_\sigma\}$, for $\sigma \in \mathfrak{S}$, span a free commutative subalgebra of $\mathbb{Q}^{\mathfrak{S}}$.*

To describe the algebraic structure in the above theorem, we introduce two new products in the vector space of permutations $\mathbb{Q}^{\mathfrak{S}}$: the super-shuffle $\underline{\sqcup}$, and the super-infiltration $\uparrow$. Both products are analogues of the classical shuffle and infiltration products for words, which we recall below.

Let A an ordered set. A **word** over A is a finite sequence of elements of A . The **length** $|w|$ of a word $w = w_1 w_2 \cdots w_n$ is the number n of letters of w . The set of all words over A is denoted by A^* , and the set of all words over A of length n is denoted by A^n . The set A^* is a monoid with the concatenation product and the empty word, denoted by λ (by abuse of notation). A word u is a **subword** of a word $w = w_1 w_2 \cdots w_n$ if there exists $i_1, i_2, \dots, i_k$ such that $u = w_{i_1} w_{i_2} \cdots w_{i_k}$. In this case we write $u \subseteq w$.

If $w, u, v \in A^*$, let

$$\binom{w}{u, v} := \left| \left\{ (x, y) : \begin{array}{l} x \subseteq w, y \subseteq w \\ x = u, y = v \\ y \text{ complementary subword of } x \text{ in } w \end{array} \right\} \right|, \quad (1)$$

and

$$\left\langle \frac{w}{u, v} \right\rangle := \left| \left\{ (x, y) : \begin{array}{l} x \subseteq w, y \subseteq w \\ x = u, y = v \end{array} \right\} \right|. \quad (2)$$

If $u \in A^n$ and $v \in A^p$, the **shuffle product** $\sqcup$ and the **infiltration product** $\uparrow$ of u and v are defined as

$$u \sqcup v := \sum_{w \in A^*} \binom{w}{u, v} w, \quad u \uparrow v := \sum_{\substack{w \in A^* \\ |w| \leq |u| + |v|}} \left\langle \frac{w}{u, v} \right\rangle w. \quad (3)$$

Example 1.

$$\begin{aligned} ba \sqcup ab &= baab + baab + baba + abab + abba + abba \\ &= 2abba + abab + 2baab + baba. \end{aligned}$$

$$\begin{aligned} ba \uparrow ab &= bab + aba + baab + baab + baba + abab + abba + abba \\ &= bab + aba + ba \sqcup ab \\ &= aba + bab + 2abba + abab + 2baab + baba. \end{aligned}$$

We remark that the infiltration product contains the shuffle product. The shuffle product was introduced by Eilenberg and MacLane in [3]. Chen, Fox and Lyndon introduced in [2] (Theorem 3.9, page 93) the infiltration product in an algebraic framework; Ochsen-schläger (see [12]) defined this product independently in the context of binomial coefficients of words. Both products are associative and commutative (see [9], proposition 6.3.12 and proposition 6.3.15).

The **standard permutation** of a word $w = a_1 \cdots a_n$ on A^* is the unique permutation, denoted by $\text{st}(w)$, such that for any $i, j \in [n]$,

$$\text{st}(w)_i < \text{st}(w)_j \iff a_i < a_j \text{ or } (a_i = a_j \text{ and } i < j).$$

The permutation $\text{st}(w)$ may be obtained by numbering from left to right the letters of w , starting with the smallest letter, then the second smallest letter, and so on. For example, if $a < b < c$ we have

$$\begin{array}{c|c} w & c b b a b b a c b \\ \hline \text{st}(w) & 8 3 4 1 5 6 2 9 7 \end{array}$$

Definition 1. If $\pi, \alpha, \beta \in \mathfrak{S}$, let

$$\left\{ \frac{\pi}{\alpha, \beta} \right\} := \left| \left\{ (x, y) : \begin{array}{l} x \subseteq \pi, y \subseteq \pi \\ \text{st}(x) = \alpha, \text{st}(y) = \beta \\ y \text{ complementary subword of } x \text{ in } \pi \end{array} \right\} \right|, \quad (4)$$

and

$$\left[\frac{\pi}{\alpha, \beta} \right] := \left| \left\{ (x, y) : \begin{array}{l} x \subseteq \pi, y \subseteq \pi \\ \text{st}(x) = \alpha, \text{st}(y) = \beta \end{array} \right\} \right|. \quad (5)$$

We are now ready to define the supershuffle and superinfiltration products:

Definition 2. Let $\alpha, \beta \in \mathfrak{S}$. The **supershuffle** and **superinfiltration** between α and β are, respectively,

$$\alpha \sqcup \beta := \sum_{\pi \in \mathfrak{S}} \left\{ \begin{matrix} \pi \\ \alpha, \beta \end{matrix} \right\} \pi, \quad \alpha \uparrow \beta := \sum_{\substack{\pi \in \mathfrak{S} \\ |\pi| \leq |\alpha| + |\beta|}} \left[\begin{matrix} \pi \\ \alpha, \beta \end{matrix} \right] \pi. \quad (6)$$

We remark that the superinfiltration product contains the supershuffle product, as the infiltration product contains the shuffle product.

Example 2.

$$21 \sqcup 1 = 213 + 132 + 2 \cdot (312 + 231) + 3 \cdot 321.$$

$$\begin{aligned} 21 \sqcup 12 &= 1243 + 3142 + 4231 + 1324 + 2413 + 2134 + 4312 + 3421 \\ &\quad + 2 \cdot (4132 + 2314 + 1423 + 3124 + 4213 + 1342 + 2431) \\ &\quad + 3 \cdot (2341 + 3214 + 1432 + 4123). \end{aligned}$$

The coefficient of 2341 in $21 \sqcup 12$ is 3 because

$$\left\{ \begin{matrix} 2341 \\ 21, 12 \end{matrix} \right\} = |\{(21, 34), (31, 24), (41, 23)\}| = 3.$$

Example 3. We have:

$$\begin{aligned} \left[\begin{matrix} 2341 \\ 231, 21 \end{matrix} \right] &= |\{(231, 21), (231, 31), (231, 41), (241, 21), \\ &\quad (241, 31), (241, 41), (341, 21), (341, 31), (341, 41)\}| = 9. \\ \left[\begin{matrix} 231 \\ 231, 21 \end{matrix} \right] &= |\{(231, 21), (231, 31)\}| = 2. \end{aligned}$$

Then:

$$231 \uparrow 21 = 231 \sqcup 21 + \cdots + 9 \cdot 2341 + \cdots + 2 \cdot 231. \quad (7)$$

Observe that $\left\{ \begin{matrix} \pi \\ \alpha, \beta \end{matrix} \right\} = \left\{ \begin{matrix} \pi \\ \beta, \alpha \end{matrix} \right\}$ and $\left[\begin{matrix} \pi \\ \alpha, \beta \end{matrix} \right] = \left[\begin{matrix} \pi \\ \beta, \alpha \end{matrix} \right]$. So, the products $\sqcup$ and $\uparrow$ are commutatives. Also, we remark that $\left\{ \begin{matrix} \pi \\ \alpha, \lambda \end{matrix} \right\} = \left[\begin{matrix} \pi \\ \alpha, \lambda \end{matrix} \right] = \left\{ \begin{matrix} \pi \\ \alpha \end{matrix} \right\}$, for all $\pi, \alpha \in \mathfrak{S}$. It is straightforward to prove that $\sqcup$ and $\uparrow$ are associatives (see [16] for more details).

Let $\mu_{\sqcup}$ and $\mu_{\uparrow}$ be the bilinear extensions of the supershuffle and superinfiltration products, respectively, and consider the map $\Delta_{\square} : \mathbb{Q}\mathfrak{S} \rightarrow \mathbb{Q}\mathfrak{S} \otimes \mathbb{Q}\mathfrak{S}$ defined by

$$\Delta_{\square}(\sigma) := \sum_{\sigma = \alpha \square \beta} \alpha \otimes \beta, \quad (8)$$

where $\square$ is defined as follow: if $\alpha = a_1 \cdots a_n \in \mathfrak{S}_n$ and $\beta = b_1 \cdots b_p \in \mathfrak{S}_p$, then

$$\alpha \square \beta := a_1 \cdots a_n (b_1 + n) \cdots (b_p + n) \in \mathfrak{S}_{n+p}.$$

Example 4. *The permutation 231456 can be decomposed as*

$$231456 = \lambda \sqcup 231456 = 231 \sqcup 123 = 2314 \sqcup 12 = 23145 \sqcup 1 = 231456 \sqcup \lambda.$$

Then:

$$\Delta_{\sqcup}(231456) = \lambda \otimes 231456 + 231 \otimes 123 + 2314 \otimes 12 + 23145 \otimes 1 + 231456 \otimes \lambda.$$

A **Hopf algebra** is a vector space that is simultaneously an associative algebra and a coassociative coalgebra, such that both structures are compatible in a certain way and that is equipped with an antiautomorphism (called the antipode) satisfying a certain property. For basic definitions and facts about Hopf algebras see [5], [7] or [15].

We have the following result:

Theorem 1. *$(\mathbb{Q}\mathfrak{S}, \mu_{\sqcup}, \Delta_{\sqcup})$ and $(\mathbb{Q}\mathfrak{S}, \mu_{\uparrow}, \Delta_{\sqcup})$ are free and cofree Hopf algebras.*

For each $\sigma \in \mathfrak{S}$, let $\varphi_{\sigma} : \mathfrak{S} \rightarrow \mathbb{Q}$ be defined by

$$\varphi_{\sigma}(\pi) := \left\{ \begin{matrix} \pi \\ \sigma \end{matrix} \right\}, \quad (9)$$

for all $\pi \in \mathfrak{S}$. We call φ_{σ} the **permutation pattern function** related to σ . It is straightforward to verify that the collection $\{\varphi_{\sigma} : \sigma \in \mathfrak{S}\}$ is linearly independent in $\mathbb{Q}^{\mathfrak{S}}$.

The product of two permutation pattern functions in $\mathbb{Q}^{\mathfrak{S}}$ can be described as follows:

Proposition 1. *Let $\alpha, \beta \in \mathfrak{S}$. The pointwise product $\varphi_{\alpha} \cdot \varphi_{\beta}$ in $\mathbb{Q}^{\mathfrak{S}}$ is given by*

$$\varphi_{\alpha} \cdot \varphi_{\beta} = \sum_{\sigma \in \mathfrak{S}} \left[\begin{matrix} \sigma \\ \alpha, \beta \end{matrix} \right] \varphi_{\sigma}. \quad (10)$$

Let $\mathcal{PPF} = \mathbb{Q}\{\varphi_{\sigma} : \sigma \in \mathfrak{S}\}$. We equip the vector space $\mathcal{PPF}$ with the coproduct

$$\Delta(\varphi_{\sigma}) := \sum_{\sigma = \alpha \sqcup \beta} \varphi_{\alpha} \otimes \varphi_{\beta}. \quad (11)$$

Theorem 2. *The vector space $\mathcal{PPF}$, with the pointwise product and the coproduct Δ , is a free and cofree Hopf algebra, isomorphic to $(\mathbb{Q}\mathfrak{S}, \mu_{\uparrow}, \Delta_{\sqcup})$ by the map $\sigma \mapsto \varphi_{\sigma}$.*

We finish this work by showing several identities and properties of permutation pattern functions, as well some occurrence of this objects in other domains. These results are expanded on and proven in the manuscript [16] of the same name.

REFERENCES

1. M. Aguiar, F. Sottile. *Structure of the Malvenuto-Reutenauer Hopf algebra of permutations*. Adv. Math. **191** no. 2, 225-275 (2005).
2. K. T. Chen, R. H. Fox, R. C. Lyndon. *Free differential calculus, IV - The quotient groups of the lower central series*, Ann. Math **65**, 163-178 (1978).
3. S. Eilenberg, S. MacLane. *On the groups $H(\Pi, n)$* , Annals of Mathematics, **58**, 55-106 (1953).
4. J.-Y. Enjalbert, H. N. Minh. *Combinatorial study of colored Hurwitz polyzêtas*. Discrete Mathematics. **312**, 3489-3497 (2012).

5. G. P. Hochschild. *Basic theory of algebraic groups and Lie algebras*. Graduate Texts in Mathematics, 75. Springer-Verlag, New York-Berlin (1981).
6. M. Hoffman. *Quasi-Shuffle products*. Journal of Algebraic Combinatorics. 11, 49-68 (2000).
7. S. A. Joni, G.C. Rota. *Coalgebras and bialgebras in combinatorics*. Umbral calculus and Hopf algebras (Norman, Okla., 1978), pp. 1-47, Contemp. Math., 6, Amer. Math. Soc., Providence, R.I., (1982)
8. D. Knuth. *The Art of Computer Programming*, Vol. 1, Fundamental Algorithms, Addison-Wesley, Reading, Massachusetts (1968).
9. M. Lothaire. *Combinatorics on words*, Encyclopedia of Mathematics, Vol 17. Addison-Wesley, Reading, MA (1983).
10. C. Malvenuto, C. Reutenauer. *Duality between quasi-symmetric functions and the Solomon descent algebra*. J. Algebra. 177, 3, 967-982 (1995).
11. K. Newman, D. Radford. *The cofree irreducible Hopf algebra on an algebra*. Amer. J. Math. 101, no. 5, 1025-1045 (1979).
12. P. Ochsenschläger. *Binomialkoeffizienten und Shuffle-Zahlen*, Technischer Bericht, Fachbereich Informatik, T. H. Darmstadt (1981).
13. S. Poirier, C. Reutenauer. *Algèbres de Hopf de tableaux*. Annales des Sciences Mathématiques du Québec 19, no. 1, 79-90 (1995).
14. D. E. Radford. *A natural ring basis for the shuffle algebra and an application to group schemes*. J. Algebra 58, 432-454 (1979).
15. C. Reutenauer. *Free Lie algebras*, The Clarendon Press Oxford University Press (1993).
16. Y. Vargas. *Hopf algebra of permutation pattern functions (complete paper)*, in preparation (2014).
17. Y. Vargas. *Hopf algebra of permutation pattern functions (extended abstract)*. Discrete Mathematics and Theoretical Computer Science, to appear (2014).

Strongly rational and algebraic permutation classes

Vince Vatter

University of Florida

At the *Permutation Patterns 2005* conference, Doron Zeilberger asked for necessary and sufficient conditions for a class to have a rational / algebraic / holonomic / ... generating function (see [2]). However, this is a stupid question, as demonstrated by the following result.

Theorem 1 (Albert, Brignall, and Vatter [1]). *Every proper permutation class is contained in a class with a rational generating function.*

(Lest I create any undeserved excitement, the proof of this result appeals to the Marcus-Tardos Theorem.)

In light of Theorem 1, it is clearly impossible to establish a sufficient hereditary condition for classes with rational (or algebraic, or holonomic, ...) generating functions. Of course, a similar issue arises in the definition of *perfect graphs*. In that context, we know that the chromatic number of a graph, $\chi(G)$, is at least its clique number, $\omega(G)$. We would like to ask which graphs achieve equality, but we're faced with the problem that given any graph, the union of this graph with a large enough complete graph will satisfy $\chi = \omega$. Therefore we say that a graph is perfect if $\chi = \omega$ for the graph and *all of its induced subgraphs*.

We shall analogously say that a permutation class is *strongly rational* (resp., *strongly algebraic*) if it and all of its subclasses have rational (resp., algebraic) generating functions. A simple counting argument proves the following.

Proposition 2. *Every strongly algebraic permutation class is wpo.*

Indeed, the following may even be true.

Conjecture 3. *A permutation class is strongly algebraic if and only if it is wpo.*

I will discuss the known and conjectured approaches to results on strongly rational and algebraic permutation classes.

References

- [1] Michael H. Albert, Robert Brignall, and Vincent Vatter. Large infinite antichains of permutations. *Pure Math. Appl. (P.U.M.A.)*, to appear.
- [2] Murray Elder and Vincent Vatter. Problems and conjectures presented at the Third International Conference on Permutation Patterns, University of Florida, March 7–11, 2005. [arXiv:math.CO/0505504](https://arxiv.org/abs/math.CO/0505504).

COMBINATORICS OF GENOCCHI NUMBERS WITH NON-COMMUTATIVE METHODS

VINCENT VONG

ABSTRACT. In this article, we are interested in generalizing the Gandhi polynomials in a non-commutative way. We show that surjective pistols arise naturally from non-commutative polynomials, which enables us to explain the combinatorial interpretation of different generalizations of the Gandhi polynomials including their q -analogs.

1. INTRODUCTION

The Genocchi numbers $(G_{2n})_{n \in \mathbb{N}}$, the Taylor coefficients of the exponential series $x \tan(\frac{x}{2})$, have been studied in several ways: first, their relation with the Euler numbers (coefficients of $\tan(x)$) has been studied by Viennot in [Vie82] who gave an overview of their combinatorial properties. But they can also be studied from one of their refinements, the Gandhi polynomials. Gandhi in [Gan70] conjectured that the sequence of polynomials defined by $C_0 = 1$ and $C_{n+1}(x) = x^2 C_n(x+1) - (x-1)^2 C_n(x)$ are polynomials such that $C_n(1) = G_{2n+2}$. This statement was proved independently by Carlitz in [Car71] and by Riordan and Stein in [RS73]. Later, these polynomials were generalized by Dumont and Foata in [DF76] as follows: $DF_1 = 1$, and $DF_{n+1}(x, y, z) = (x+z)(y+z)DF_n(x, y, z+1) - z^2 DF_n$. In order to give a combinatorial interpretation of these, they introduced the surjective pistols that are one of the combinatorial objects enumerated by the Genocchi numbers. They showed that the different parameters x , y , and z correspond to different statistics on surjective pistols. Another interesting property of these polynomials is that they are symmetric in the three variables x , y , z . This last property has been proved by Dumont and Foata in [DF76], and Carlitz gave an explicit symmetric formula in [Car80]. Han in [Han96] gave another combinatorial interpretation of the Dumont-Foata polynomials, and found back their interpretation. Then Dumont gave a generalization with six parameters. Independently, Zeng and Randrianarivony respectively in [Zen96] and [Ran94] provided different properties about these. Finally, Han and Zeng in [HZ99] showed that the Gandhi polynomials admit a q -analog, and provided several combinatorial interpretations of their coefficients.

In this paper, we use the non-commutative paradigm in order to find back that the combinatorial interpretation of Gandhi polynomials and to find back a new interpretation of the q -analog defined by Han and Zeng.

Key words and phrases. Genocchi numbers, surjective pistols, non-commutative polynomials.

2. THE GANDHI POLYNOMIALS

Let us define the Gandhi polynomials, denoted by $(C_n)_{n \geq 1}$, as follows:

$$(1) \quad \begin{cases} C_1(x) &= 1 \\ C_{n+1}(x) &= \Delta(C_n(x) \cdot x^2) \text{ if } n \geq 1, \end{cases}$$

where $\Delta(f(x)) = f(x+1) - f(x)$, for any function f .

It is known ([DF76]) that these polynomials have a combinatorial interpretation in terms of surjective pistols, which are defined by:

Definition 2.0.1. A *surjective pistol* p of size $2n$ is a surjective map from $\{1, \dots, 2n\}$ onto $\{2, \dots, 2n\}$ such that for each i in $\{1, \dots, 2n\}$, $p(i) \geq i$.

The set of surjective pistols of size $2n$ is denoted by $\mathcal{P}_{2n}$. From now on, a surjective pistol p of size $2n$ is represented by a word w of size $2n$ such that the i -th letter of w is the value $p(i)$. For example, the word $w = 226466$ represents the surjective pistol of size 6 which sends 1 to 2, 2 to 2, 3 to 6, 4 to 4, 5 to 6 and 6 to 6.

Note that from a surjective pistol p of size $2n$, we obtain a surjective pistol of size $2n+2$ by concatenating twice $2n+2$ to the right of p . This operation seems intuitively related to the multiplication by x^2 on the induction formula of Gandhi polynomials (1). But we cannot obtain all surjective pistols of size $2n+2$ by this process. But in the word $p \cdot (2n+2) \cdot (2n+2)$, if we substitute some but not all values $2n$ by $2n+2$, we have still a surjective pistol. For example, if $p = 226466$, then 22646688, 22846688, 22648688, 22646888, 22848688, 22846888 and 22648888 are indeed surjective pistols. The substitution process looks intuitively like an analog of the finite difference on words. Indeed, as we shall see, the whole set $\mathcal{P}_{2n+2}$ is generated in this way, each surjective pistol exactly once.

The non-commutative polynomials will help us formalize the previous ideas. We then proceed as follows: we build a non-commutative analog of the Gandhi polynomials, and show how surjective pistols arise naturally from these.

Definition 2.0.2. Let A be an alphabet. The R -algebra of *non-commutative polynomials* over A , denoted by $\mathbb{R}\langle A \rangle$, is the algebra generated by the letters of A with the concatenation product. So a linear basis is given by the finite words on A .

From now on, our alphabet is $A = \{a_i, \text{ for } i \in \mathbb{N}^*\} \cup \{a_\infty\}$, and $R = \mathbb{R}$. Let us give a sense to "having a non-commutative analog". There is a natural projection Π from $\mathbb{R}\langle A \rangle$ onto $\mathbb{R}[x]$:

$$\begin{cases} \Pi(a_\infty) &= x, \\ \Pi(a_i) &= 1 \\ \Pi(w_1 \cdots w_n) &= \Pi(w_1) \cdots \Pi(w_n) \end{cases} \quad \text{for } i \in \mathbb{N}^*$$

Then a non-commutative analog of a polynomial P is an element $\mathbf{P}$ of $\mathbb{R}\langle A \rangle$ such that $\Pi(\mathbf{P})$ is equal to P . In the same way, if L is a linear operator on $\mathbb{R}[x]$, we say that $\mathbf{L}$ is a non-commutative analog of L , if $\Pi \circ \mathbf{L}$ is equal to $L \circ \Pi$. Since the Gandhi polynomials are built from the operator Δ , one way to have non-commutative Gandhi polynomials is to construct these from non-commutative analogs of Δ . Since the operator Δ is constructed from the operator consisting in shifting by 1 (denoted by T), it is natural to build first the non-commutative analog of T . Let

us denote by T_i the algebra homomorphism such that:

$$\begin{cases} T_i(a_\infty) &= a_\infty + a_i, \\ T_i(a_j) &= a_j, \end{cases} \quad \text{for } j \text{ in } \mathbb{N}^*.$$

For each positive integer i , we have indeed $T \circ \Pi = \Pi \circ T_i$. Then, we define Δ_i by $\Delta_i = T_i - Id_{\mathbb{R}\langle A \rangle}$.

Example 1. For example, if $w = a_2 a_2 a_\infty a_4 a_\infty a_\infty$, then we have:

$$(2) \quad \begin{aligned} \Delta_6(w) &= a_2 a_2 a_6 a_4 a_\infty a_\infty + a_2 a_2 a_\infty a_4 a_6 a_\infty + a_2 a_2 a_\infty a_4 a_\infty a_6 \\ &\quad + a_2 a_2 a_6 a_4 a_6 a_\infty + a_2 a_2 a_\infty a_4 a_6 a_6 + a_2 a_2 a_6 a_4 a_\infty a_6 + a_2 a_2 a_6 a_4 a_6 a_6. \end{aligned}$$

Note that for a word w , the terms in $\Delta_i(w)$ are exactly the words obtained by replacing at least one of the a_∞ of w by an a_i . Now, let us define the non-commutative Gandhi polynomials as follows:

$$(3) \quad \begin{cases} C_1 &= 1 \\ C_{n+1} &= \Delta_{2n}(C_n a_\infty a_\infty) \quad \text{if } n > 1. \end{cases}$$

In particular, we have:

$$(4) \quad \begin{aligned} C_2 &= a_2 a_2 + a_2 a_\infty + a_\infty a_2 \\ C_3 &= a_2 a_2 a_4 a_4 + a_2 a_2 a_\infty a_4 + a_2 a_2 a_4 a_\infty + a_2 a_4 a_4 a_4 + a_2 a_\infty a_4 a_4 \\ &\quad + a_2 a_4 a_\infty a_4 + a_2 a_4 a_4 a_\infty + a_2 a_4 a_\infty a_\infty + a_2 a_\infty a_4 a_\infty \\ &\quad + a_2 a_\infty a_\infty a_4 + a_4 a_2 a_4 a_4 + a_\infty a_2 a_4 a_4 + a_4 a_2 a_\infty a_4 \\ &\quad + a_4 a_2 a_4 a_\infty + a_4 a_2 a_\infty a_\infty + a_\infty a_2 a_4 a_\infty + a_\infty a_2 a_\infty a_4. \end{aligned}$$

Now, surjective pistols of positive size appear naturally through this definition. Indeed, let us define the embedding i from surjective pistols of positive size onto $R\langle A \rangle$. If p is a surjective pistol of size $2n$ (with $n \geq 1$), then $i(p)$ is equal to the word w of size $2n-2$ such that the i -th letter of w is a_∞ if $p(i)$ is equal to $2n$, and is $a_{p(i)}$ otherwise. For example, if $p = 22646688$, then $i(p) = a_2 a_2 a_6 a_4 a_6 a_6$. If $p = 22$, then $i(p) = \epsilon$, the empty word, and if $p = 28648688$, then $i(p) = a_2 a_\infty a_6 a_4 a_\infty a_6$. The map i is clearly an injective map: let w be a word of size $2n$ of the image of i we obtain the corresponding surjective pistol by replacing the a_∞ by a_{2n+2} , by reading the indices of the new word, and by adding twice to the right the letter $2n+2$. In the sequel, we denote by $\mathbf{p}$ the word $i(p)$. We have

Proposition 2.1. *Let $n \geq 1$. Then:*

$$(5) \quad C_n = \sum_{p \in \mathcal{P}_{2n}} \mathbf{p}.$$

Proof. By induction on n . The statement is satisfied for $n = 1$. Assume that it is true for $n \geq 1$. By induction hypothesis on C_n , we have:

$$\begin{aligned} C_{n+1} &= \Delta_{2n}(C_n a_\infty a_\infty) \\ &= \Delta_{2n}(\sum_{p \in \mathcal{P}_{2n}} \mathbf{p} a_\infty a_\infty) \\ &= \sum_{p \in \mathcal{P}_{2n}} \Delta_{2n}(\mathbf{p} a_\infty a_\infty). \end{aligned}$$

Let p be a surjective pistol of size $2n$. Then the elements of $\Delta_{2n}(\mathbf{p} a_\infty a_\infty)$ are the words associated with surjective pistols of size $2n+2$: they are words $w = w_1 \cdots w_{2n}$ of size $2n$, such that at least one a_∞ of $\mathbf{p} a_\infty a_\infty$ is replaced by a_{2n} , and such that $w_i = a_{p(i)}$ if $p(i) \leq 2n-2$, and $w_i = a_{2n}$ or a_∞ otherwise. They are then associated with surjective pistols p' of size $2n+2$, where for $i \leq 2n$, if $p(i) \leq 2n-2$, then $p'(i) = p(i)$, and $p'(i) = 2n$ or $2n+2$ otherwise. Conversely, if p' is a surjective pistol

of size $2n+2$ such that $p'(i) = p(i)$ if $p(i) < 2n$, and $p'(i) = 2n$ or $p'(i) = 2n+2$ otherwise, then $\mathbf{p}'$ is a term of $\Delta_{2n}(\mathbf{p} a_{\infty} a_{\infty})$.

Let us denote by $\mathcal{D}_p$ the set of pistols p' such that $\mathbf{p}'$ is a term of $\Delta_{2n}(\mathbf{p} a_{\infty} a_{\infty})$. By the previous observation, for p in $\mathcal{P}_{2n}$ it is in fact the following set:

$$(6) \quad \mathcal{D}_p = \left\{ p' \in \mathcal{P}_{2n+2} \mid \begin{array}{ll} p'(i) = p(i) & \text{if } i < 2n+1 \text{ and } p(i) < 2n, \\ p'(i) = 2n & \text{or } 2n+2 \text{ otherwise} \end{array} \right\}.$$

We have: $\mathcal{P}_{2n+2} = \sqcup_{p \in \mathcal{P}_{2n}} \mathcal{D}_p$. Indeed, let p' be in $\mathcal{P}_{2n+2}$. By deleting the last two letters and replacing the $2n+2$ by $2n$, we obtain a surjective pistol p of size $2n$, and p' is in $\mathcal{D}_p$. So we have $\mathcal{P}_{2n+2} = \cup_{p \in \mathcal{P}_{2n}} \mathcal{D}_p$. Let p and p' be two different surjective pistols of size $2n$. Then, there exists an i such that $p(i)$ is different from $p'(i)$. By symmetry, we assume that $p(i) < p'(i) \leq 2n$. Then for each element q in $\mathcal{D}_p$, we have $q(i) = p(i)$, and for each q' in $\mathcal{D}_{p'}$, we have $q'(i) \geq p'(i)$. In particular, $q(i) < q'(i)$, so $\mathcal{D}_p \cap \mathcal{D}_{p'}$ is empty. We deduce that:

$$(7) \quad \begin{aligned} C_{n+1} &= \sum_{p \in \mathcal{P}_{2n}} \Delta_{2n}(\mathbf{p} a_{\infty} a_{\infty}) \\ &= \sum_{p \in \mathcal{P}_{2n}} \sum_{p' \in \mathcal{D}_p} \mathbf{p}' \\ &= \sum_{p \in \mathcal{P}_{2n+2}} \mathbf{p}. \end{aligned}$$

□

3. THE Q-PROJECTION AND THE Q-GANDHI POLYNOMIALS

The q -Gandhi polynomials are defined as follows:

$$(8) \quad \begin{cases} C_1(x, q) &= 1 \\ C_{n+1}(x, q) &= \Delta_q(C_n x^2) \text{ for } n \geq 1. \end{cases}$$

Since we have defined non-commutative Gandhi polynomials, in order to obtain C_n , we have to find a q -version Π_q of the map Π , such that $\Pi_q(C_n) = C_n$. By induction and linearity, it is enough to find Π_q such that for p a surjective pistol of size $2n$, we have:

$$(9) \quad \Pi_q \circ \Delta_{2n}(\mathbf{p} a_{\infty} a_{\infty}) = \Delta_q(\Pi_q(\mathbf{p}) x^2).$$

From the previous remark, it is enough to define Π_q on words w of the form $w = \mathbf{p}$ and $w = \mathbf{p} a_{\infty} a_{\infty}$ where $\mathbf{p}$ is associated with a surjective pistol. Equation (9) indicates that we can fix a surjective pistol p , and work on $\mathcal{D}_p$. Let p be a surjective pistol, we look for Π_q such that: $\Pi_q(\mathbf{p}) = q^{\text{stat}(p)} \Pi(\mathbf{p})$, and $\Pi_q(\mathbf{p} a_{\infty} a_{\infty}) = q^{\text{stat}(p)} \Pi(\mathbf{p}) x^2$. Now, we have to find a function stat . One way to proceed is to note that $\Pi(\mathbf{p}) = x^{\text{maj}(p)}$. Then, we compute the two members of (9), which gives condition on stat . In another context, Novelli, Thibon and Williams in [NTW10] defined the statistic that we use. It is closely related to the statistic defined by Han and Zeng in [HZ99].

Definition 3.0.3. Let w be a word of size $2n$ containing $a_2, \dots, a_{2n-2}$. Let $j_2, \dots, j_{2n-2}$ be the positions of the last occurrence of $a_2, \dots, a_{2n-2}$. Then a *special inversion* of w is a pair (i, j) with $i < j$ such that j is one of the j_k and $w_i > w_j$. We denote by $\text{sinu}(w, a_j)$ the number of special inversions of w such that the letter corresponding to the second coordinate is a_j , and by $\text{sinu}(w)$ the number of special inversions of w .

Example 2. If $w = a_4 a_2 a_6 a_4 a_{\infty} a_6$, we have $\text{sinu}(w, a_2) = 1$, $\text{sinu}(w, a_4) = 1$, $\text{sinu}(w, a_6) = 1$, and $\text{sinu}(w) = 3$.

Definition 3.0.4. Let p be a surjective pistol. We set:

$$(10) \quad \begin{aligned} \Pi_q(\mathbf{p}) &= q^{\text{snv}(p)} \Pi(\mathbf{p}) = q^{\text{snv}(p)} x^{\text{max}(p)}, \\ \Pi_q(\mathbf{p} a_\infty a_\infty) &= q^{\text{snv}(p)} \Pi(\mathbf{p} a_\infty a_\infty) = q^{\text{snv}(p)} x^{\text{max}(p)+2}. \end{aligned}$$

Proposition 3.1. *With the previous definitions, Equation (9) is satisfied.*

The following proposition helps us to prove 3.1.

Proposition 3.2. *Let p be a surjective pistol of size $2n$, and l be the number of a_∞ in $\mathbf{p} a_\infty a_\infty$. Then $\Pi(\mathbf{p} a_\infty a_\infty) = x^l$, and so $\Delta_q(\Pi(w))$ is equal to:*

$$(11) \quad \sum_{i=0}^{l-1} \left(\sum_{k=i}^{l-1} \binom{k}{i} q^{k-i} \right) x^{l-1-i}.$$

We also have:

$$(12) \quad \Pi \left(\sum_{p' \in \mathcal{D}_p} q^{\text{snv}(p', a_{2n})} \mathbf{p}' \right) = \sum_{i=0}^{l-1} \left(\sum_{k=i}^{l-1} \binom{k}{i} q^{k-i} \right) x^{l-1-i}.$$

Proof. First, we regroup the elements of $\mathcal{D}_p$ according to the number of occurrences of a_∞ , and by the value $\text{snv}(\cdot, a_{2n})$. Thus, two surjective pistols p' and p'' are in the same class if and only if they have same number of a_∞ and $\text{snv}(p', a_{2n}) = \text{snv}(p'', a_{2n})$. Since in $\mathbf{p}'$ and $\mathbf{p}''$ only the letter a_∞ is greater than a_{2n} , we deduce that the position of the rightmost a_{2n} of $\mathbf{p}'$ and $\mathbf{p}''$ are the same. Note that if one p' in $\mathcal{D}_p$ has $l-i-1$ letters equal to a_∞ , then the value $\text{snv}(p', a_{2n})$ is between 0 and $l-i-1$. So,

$$(13) \quad \sum_{p' \in \mathcal{D}_p} \mathbf{p}' = \sum_{i=0}^{l-1} \sum_{k=i}^{l-1} \sum_{\substack{p' \in \mathcal{D}_p \\ \text{oc}(\mathbf{p}', a_\infty) = l-1-i \\ \text{snv}(\mathbf{p}', a_{2n}) = k-i}} q^{k-i} \mathbf{p}',$$

where $\text{oc}(\mathbf{p}', a_\infty)$ is the number of a_∞ in $\mathbf{p}'$. Let us count one equivalence class. If the elements $\mathbf{p}'$ have $l-i-1$ letters equal a_∞ , and $\text{snv}(\mathbf{p}', a_{2n})$ is equal to $k-i$, then the rightmost a_{2n} is at the same position j for all elements of the class. To the left of j , we know that there are $k-i$ letters equal to a_∞ ($\text{snv}(\mathbf{p}', a_{2n}) = k-i$) and i letters equal to a_{2n} (j is the position of the rightmost a_{2n} and $\text{oc}(\mathbf{p}', a_\infty) = m-i-1$). Then this class has $\binom{k}{i}$ elements. By applying Π to (13), we find (12). $\square$

In [IIZ99], Han and Zeng have defined the following generalization of Gandhi polynomials:

$$(14) \quad D_{n+1}(x) = \begin{cases} 1 & \text{if } n = 0, \\ \Delta_q(D_n(x)x^2) + (y-1)D_n(x)x & \text{if } n \geq 1. \end{cases}$$

They also give some combinatorial interpretations of these polynomials. With our non-commutative method, we see that the parameter y corresponds to non double fixed point, and another q -statistic arises naturally, and is snv , the number of special non inversions, which are pairs (i, j) such that $i < j \leq k$, and $w_i < w_j$, where $w_i = a_k$.

Example 3. If $w = a_4 a_2 a_6 a_4 a_\infty a_6$, then $\text{snv}(w, a_2) = 0$, $\text{snv}(w, a_4) = 1$, $\text{snv}(w, a_6) = 1$, and $\text{snv}(w) = 2$.

In the same way as before, we can get the following proposition:

Proposition 3.3. *Let p be a surjective pistol and define $\Pi_q(\mathbf{p}) = q^{\text{snv}(p)}\Pi(\mathbf{p})$. Then, we have:*

$$(15) \quad \Pi_q \left(\sum_{p \in \mathcal{P}_{2n}} y^{\text{ndfix}(p)} \mathbf{p} \right) = D_n(x).$$

REFERENCES

- [Car71] L. Carlitz. A conjecture concerning Genocchi numbers. *Norske Vid. Selsk. Skr. (Trondheim)*, (9):4, 1971.
- [Car80] L. Carlitz. Explicit formulas for the Dumont-Foata polynomial. *Discrete Math.*, 30:211–225, 1980.
- [DF76] D. Dumont and D. Foata. Une propriété de symétrie des nombres de Genocchi. *Bull. Soc. Math. Fr.*, 104:433–451, 1976.
- [Gan70] J.M. Gandhi. A conjectured representation of Genocchi numbers. *Am. Math. Mon.*, 77:505–506, 1970.
- [Han96] G.-N. Han. Symétries trivariées sur les nombres de Genocchi. *Eur. J. Comb.*, 17(4):397–407, 1996.
- [HZ99] G.-N. Han and J. Zeng. q -polynômes de Gandhi et statistique de Denert. *Discrete Math.*, 205(1-3):119–143, 1999.
- [NTW10] J.-C. Novelli, J.-Y. Thibon, and L.K. Williams. Combinatorial hopf algebras, non-commutative hall–littlewood functions, and permutation tableaux. *Adv. in Math.*, 224(4):1311 – 1348, 2010.
- [Ran94] A. Randrianarivony. Polynômes de Dumont-Foata généralisés. *Sémin. Lothar. Comb.*, 32:12, 1994.
- [RS73] J. Riordan and P.R. Stein. Proof of a conjecture on genocchi numbers. *Discrete Math.*, 5(4):381–388, 1973.
- [Vie82] G. Viennot. Interprétations combinatoires des nombres d’Euler et de Genocchi. *Sémin. Théor. Nombres, Univ. Bordeaux I*, 1981-1982:94, 1982.
- [Zen96] J. Zeng. Sur quelques propriétés de symétrie des nombres de Genocchi. *Discrete Math.*, 153(1-3):319–333, 1996.

LABORATOIRE D’INFORMATIQUE GASPARD MONGE, UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE,
5 BOULEVARD DESCARTES, CHAMPS-SUR-MARNE, 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE

PATTERN AVOIDANCE ON CLANS, SKEW PARTITIONS, AND PAIRS OF 2-BLOCK SET PARTITIONS

ALEXANDER WOO

ABSTRACT. A clan is a partial matching (on a fixed number of linearly ordered vertices) with the additional data of a sign on each unmatched vertex. There is a natural notion of pattern containment and avoidance on clans. We describe a bijection, motivated by algebraic geometry, between non-crossing clans and pairs of permutations with one ascent/descent, which can be viewed as ordered 2-block set partitions. It is easiest to see this bijection through intermediate bijections with skew partitions. We use it to translate an interval pattern avoidance condition on pairs of permutations (which is a kind of restricted avoidance on pairs) to a pattern avoidance condition on clans.

This talk will be based on joint work with Ben Wyser, specifically the paper [WW14].

Let p and q be positive integers, and $n = p + q$. A (p, q) -**clan** is a partial matching of vertices implicitly labeled $\{1, 2, \dots, n\}$ with unpaired vertices assigned a $+$ or a $-$. In addition, the number of matchings and the number of $+$'s should add up to p (or, equivalently, the number of matchings and the number of $-$'s should add up to q). We will draw a clan

like $+ \begin{array}{c} \diagup \\ \diagdown \end{array}$, which is the $(2, 2)$ -clan with a $+$ at position 1, a $-$ at position 3, and positions 2 and 4 matched.

There is a natural notion of pattern containment on clans. If δ is a (p, q) -clan and γ is a (r, s) -clan, with $p \leq r$ and $q \leq s$, then $\delta \subseteq \gamma$ if there exist indices $1 \leq i_1 < \dots < i_{p+q} \leq r + s$ satisfying all of the following:

- i_j and i_k are matched in γ if and only if j and k are matched in δ .
- i_j is a $+$ in γ if and only if j is a $+$ in δ .
- i_j is a $-$ in γ if and only if j is a $-$ in δ .

A **non-crossing** clan is one that avoids $\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$.

For each clan γ , there is an associated algebraic variety Y_γ . We are interested in when Y_γ satisfies various geometric properties. For example, we want to characterize the clans γ for which Y_γ is smooth or lci. In the case where γ is non-crossing, there are permutations $u(\gamma)$ and $v(\gamma)$ such that $Y_\gamma = X_{u(\gamma)}^{v(\gamma)}$, where $X_{u(\gamma)}^{v(\gamma)}$ is another algebraic variety known as a Richardson variety [Wys13]. There is a known characterization of whether X_u^v is smooth. (For lci, the characterization is conjectural, but enough of the conjecture has been proven for our purposes.)

At a sufficiently snooty algebraic geometry conference¹, there would be nothing else to say. For Permutation Patterns, the interesting part is yet to come.

AW is partially supported by NSA Young Investigators Grant H98230-13-1-0242.

¹To be fair to algebraic geometers, they have gotten much less snooty over the years, so much so that such conferences may constitute the empty set.

First of all, the characterization of when $X_{u(\gamma)}^{v(\gamma)}$ is smooth (or lci) uses a generalization of pattern avoidance called **interval pattern avoidance**. A pair of permutations $[x, w]$ contains a second pair $[u, v]$ if:

- x contains u and w contains v in exactly the same locations
- x and w have the same entries outside those locations
- $\text{inv}(w) - \text{inv}(x) = \text{inv}(v) - \text{inv}(u)$, where inv denotes the number of inversions.

The last condition can be translated into a mesh avoidance condition.

To state the characterization of smooth X_v^u , one also needs **Bruhat order** on permutations. If σ is a permutation and t a transposition such that $\text{inv}(t\sigma) \geq \text{inv}(\sigma)$, then declare $t\sigma \geq \sigma$ in Bruhat order, and Bruhat order is the smallest partial order on S_n including these relations (or, in other words, it is the transitive closure of this relation).

We now have the following theorem on when X_τ^σ is smooth, combining results of [KWY13, BW03, Cor03, KLR03, Man01].

Theorem 1. *Let S be the set of pairs of permutations including all of the following:*

- $[(a+2) \cdots 1(a+b+4) \cdots (a+3), (a+b+4)(a+2) \cdots 2(a+b+3) \cdots (a+3)1]$ where a and b are nonnegative integers
- $[(a+1) \cdots 1(a+3)(a+2)(a+b+4) \cdots (a+4), (a+3)(a+1) \cdots 2(a+b+4)1(a+b+3) \cdots (a+4)(a+2)]$ where a and b are nonnegative integers
- $[1(c+3) \cdots 2(c+4), (c+3)(c+4)(c+2) \cdots 312]$ where c is a nonnegative integer.
(The case $c = 0$ is the same as previous case with $a = b = 0$.)

Then the variety X_τ^σ is NOT smooth if and only if one (or both) of the following occurs:

- *There exists ρ with $\tau \leq \rho \leq \sigma$ such that $[\rho, \sigma]$ contains a pair in S .*
- *There exists ρ with $\sigma^c \leq \rho \leq \tau^c$ such that $[\rho, \tau^c]$ contains a pair in S .*

It is known that a similar characterization exists for when X_τ^σ is lci (and indeed for any reasonable local geometric property), but the precise set S is not known, though there is a conjecture. Enough of this conjecture has been proven for our current purposes.

Second, for any non-crossing clan γ , the permutations $v(\gamma)$ and $u(\gamma)$ respectively have at most one ascent and one descent, always between entries p and $p+1$. This means they can really be thought of as 2-block ordered set partitions $\mathbf{u}$ and $\mathbf{v}$ with blocks of sizes p and q . There is a natural way of associating a partition $\lambda(\sigma)$ to any 2-block ordered set partition σ , as follows.

Given a 2-block ordered set partition τ , define the partition $\lambda(\tau)$ by drawing its outer boundary (in French notation) as follows. At the i -th step, go right if i is in the second block, and down if i is in the first block. Alternatively, $\lambda(\tau)_j$ is defined to be the number of entries in the second block of τ strictly less than $p+1-j$.

This bijection between 2-block ordered set partitions and skew partitions translates Bruhat order to containment. If $\tau \leq \sigma$ in Bruhat order, then $\lambda(\tau) \subseteq \lambda(\sigma)$ (by which I mean containment of partition diagrams). Hence we can think of $\lambda(\sigma)/\lambda(\tau)$ as a skew partition.

We can then translate the interval pattern avoidance conditions above to conditions on the skew partition. When σ has at most one descent and τ at most one ascent, both at the same location, only the first kind of pairs in S are relevant. In this case, the pattern avoidance condition translates to the following:

Theorem 2. *Suppose σ has at most one ascent, at p , and τ has at most one descent, at p . Then X_τ^σ is NOT smooth if and only if one (or both) of the following occurs:*

- The outer boundary of $\lambda(\sigma)$ has $a + 1$ down moves followed by $b + 1$ right moves, and the outer boundary of $\lambda(\tau)$ does not touch the boundary of $\lambda(\sigma)$ at the corner between the down moves and the right moves.
- The outer boundary of $\lambda(\tau)$ has $a + 1$ right moves followed by $b + 1$ down moves, and the outer boundary of $\lambda(\sigma)$ does not touch the boundary of $\lambda(\tau)$ at the corner between the right moves and down moves.

Here the a and b correspond with the a and b in the first set of pairs of S , and the two conditions correspond respectively to the conditions on $\tau \leq \rho \leq \sigma$ and on $\sigma^c \leq \rho \leq \tau^c$. As you can see, a and b are actually irrelevant; X_τ^σ is not smooth when the skew partition has what we call an **inner corner**. However, a and b are relevant for more subtle properties such as being lci.

Finally, we translate the conditions on the skew partition to a pattern avoidance condition on the clans. One can describe the bijection between non-crossing clans and skew partitions specified by the geometry as follows.

- On the opening of a matching, the upper partition boundary goes right and the lower partition boundary goes down
- On the closing of a matching, the upper partition boundary goes down and the lower partition boundary goes right
- On a $+$, both partition boundaries go down
- On a $-$, both partition boundaries go right

Translating the above conditions on smoothness using skew partitions to clans, we have

Theorem 3. *Let γ be a non-crossing clan. Then Y_γ is NOT smooth if and only if γ contains one of the following 8 clans: $\overbrace{+-}^{\frown}$, $\overbrace{-+}^{\frown}$, $\overbrace{+\frown}^{\frown}$, $\overbrace{-\frown}^{\frown}$, $\overbrace{\frown+}^{\frown}$, $\overbrace{\frown-}^{\frown}$, and $\overbrace{\frown\frown}^{\frown}$.*

A similar theorem is true for lci, involving avoiding 35 clans.

Interestingly, for another geometric property, the property of being Gorenstein, ordinary pattern avoidance works neither on patterns nor on clans. One way to see this is in terms of the skew partition. A skew partition is NOT Gorenstein if and only if it has a corner with a preceding down moves and b succeeding up moves where $a \neq b$. It is interesting that this is almost the canonical example of a non-regular language.

REFERENCES

- [BW03] S. Billey, G. Warrington, *Maximal singular loci of Schubert varieties in $SL(n)/B$* , Trans. Amer. Math. Soc. **335** (2003) 3915–3945.
- [Cor03] A. Cortez, *Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire*, Adv. Math. **178** (2003) 396–445.
- [KLR03] C. Kassel, A. Lascoux, C. Reutenauer, *The singular locus of a Schubert variety*, J. Algebra **269** (2003) 74–108.
- [KWY13] A. Knutson, A. Woo, and A. Yong, *Singularities of Richardson varieties*, Math. Res. Lett. **20** (2013) 391–400.
- [Man01] L. Manivel, *Le lieu singulier des variétés de Schubert*, Int. Math. Res. Not. **16** (2001) 849–871.
- [WW14] A. Woo and B. Wyser, *Combinatorial results on $(1, 2, 1, 2)$ -avoiding $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ -orbit closures on $GL(p + q, \mathbb{C})/B$* , preprint available as ArXiv:1403.0363.
- [Wys13] B. Wyser, *Schubert calculus of Richardson varieties stable under spherical Levi subgroups*, J. Algebraic Combin. **38** (2013) 829–850.

Pattern Avoidance in Ordered Set Partitions

Ruyue (Julia) Yuan, Valparaiso University

In this poster, we consider the enumeration of ordered set partitions avoiding a permutation pattern. Recently, many results have been published concerning this topic, including enumerative results by Bruner, Chen et al., Godbole et al., and Kasraoui. We specifically consider Kasraoui's summation formula for the number of ordered set partitions that avoid a pattern of length 3, and analyze it in a variety of ways. We focus on the case where the number of parts is comparable to n , and, through parameterization, we find the value of i for which the maximum value in the sum occurs. We develop a simplified approximation of the formula, and we then obtain a lower bound for the resulting sum. All results are thus asymptotic as the number of parts in the partition is allowed to grow to infinity with n . This is joint work with Stephanie Degraaf, Kai Orans, and Anant Godbole