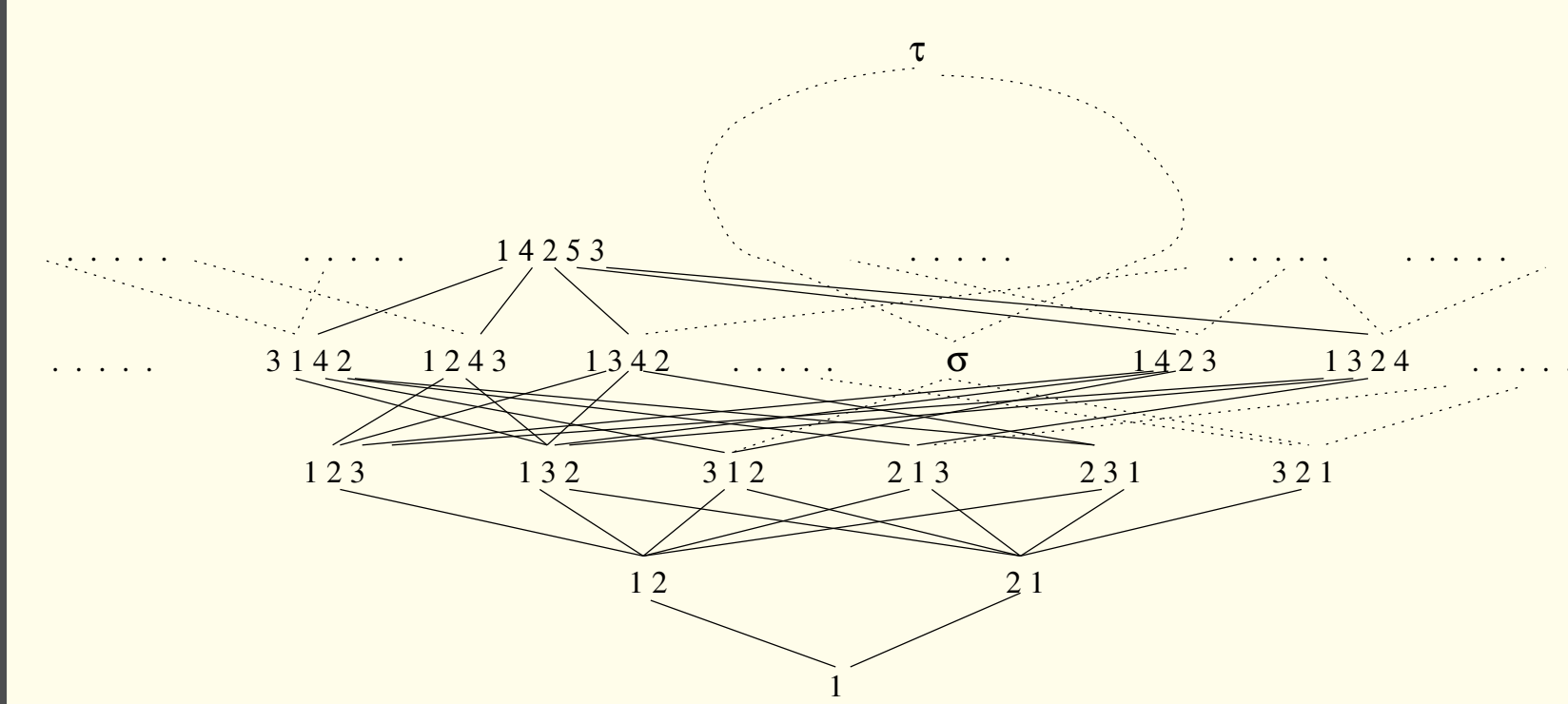


On the Möbius function of the quasi-consecutive pattern poset

Antonio Bernini, Luca Ferrari
antonio.bernini@unifi.it, luca.ferrari@unifi.it



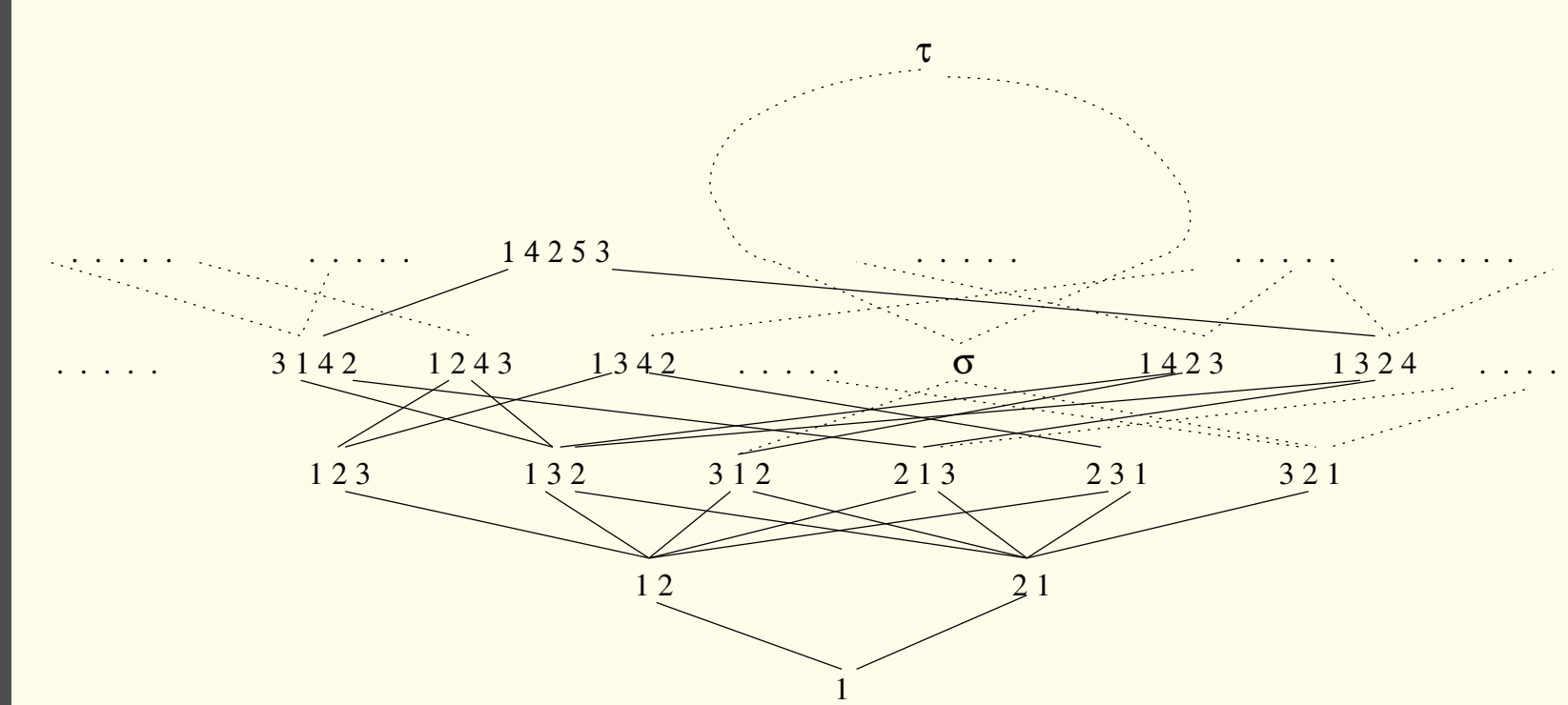
The pattern poset



- $\leq \rightsquigarrow$ pattern containment;
- $2-3-1 \leq 14253$;
- $\mu(\sigma, \tau) = ?$ partial results

Sagan and Vatter (2006), Steingrímsson and Tenner (2010), Burstein, Jelínek, Jelínková and Steingrímsson (2011) and McNamara and Steingrímsson (2013).

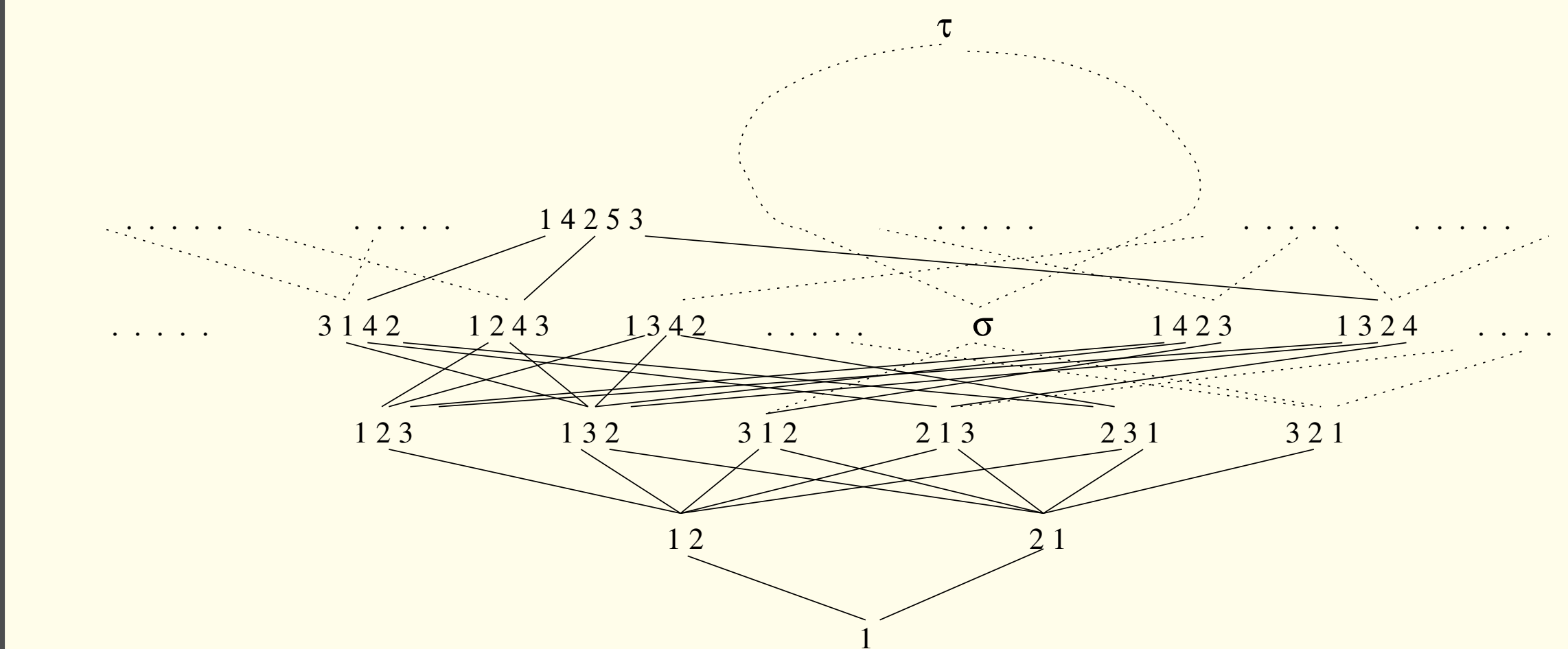
The consecutive pattern poset



- $\leq \rightsquigarrow$ consecutive pattern containment;
- $231 \not\leq 14253$; $132 \leq 14253$;
- $\mu(\sigma, \tau)$: completely understood

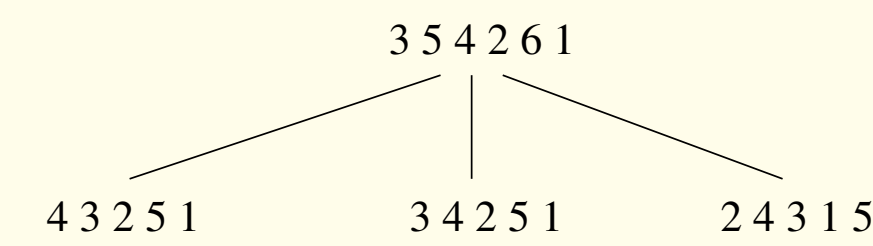
B., Ferrari and Steingrímsson (2011), Sagan and Willenbring (2012).

The quasi-consecutive pattern poset



In the quasi-consecutive pattern poset, $\sigma \leq \tau$ when there is an occurrence of σ in τ whose first two elements are possibly not consecutive, whereas all the remaining elements has to appear consecutively in τ .

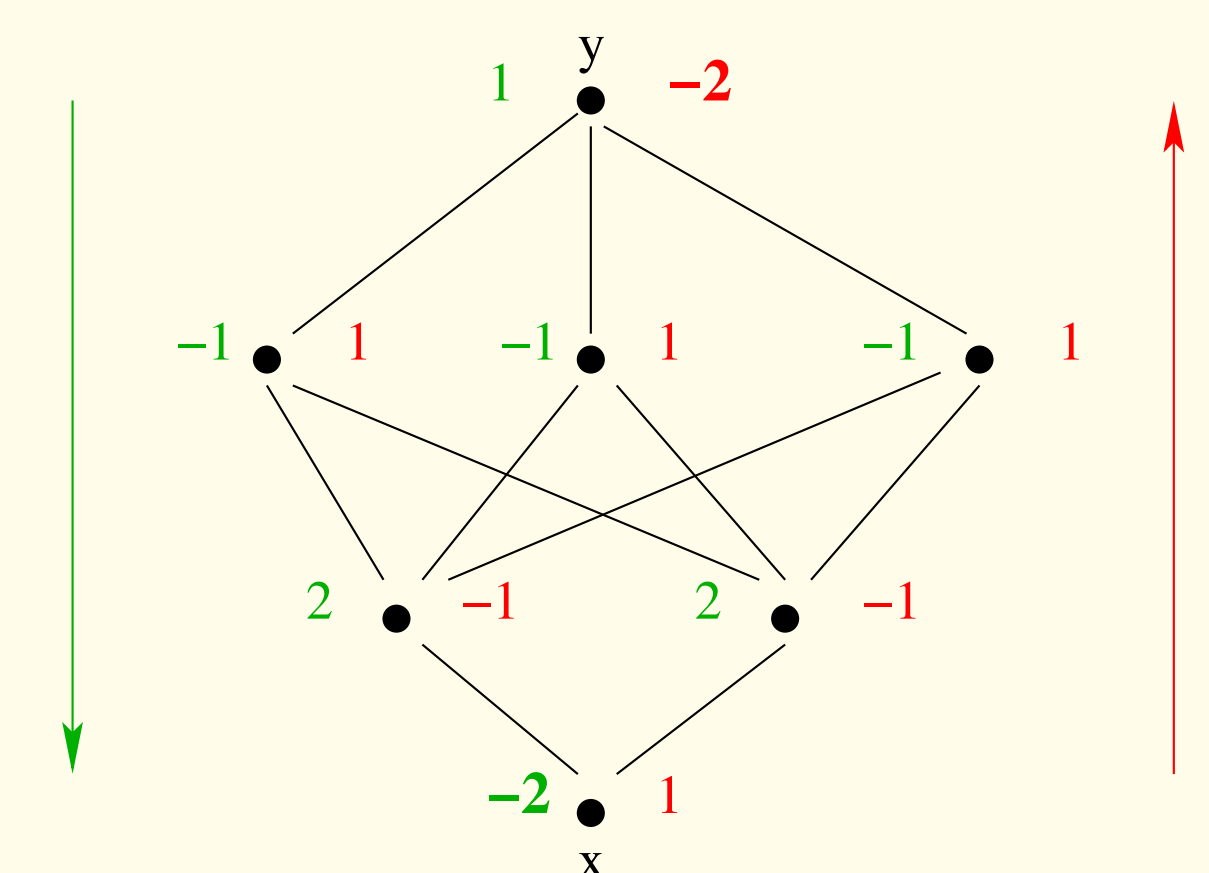
A permutation τ covers at most three permutations



obtained by deleting either the first, the second or the last element of τ (and renaming the remaining ones to get a permutation). Two or three of the covered permutations may coincide.

- $\leq \rightsquigarrow$ quasi-consecutive pattern containment
- $2-31 \leq 14253$; $3-12 \not\leq 14253$
- $\mu(\sigma, \tau)$ completely understood in simple general cases and in the case of only one occurrence of σ in τ

Möbius function



$$\mu(x, x) = 1$$

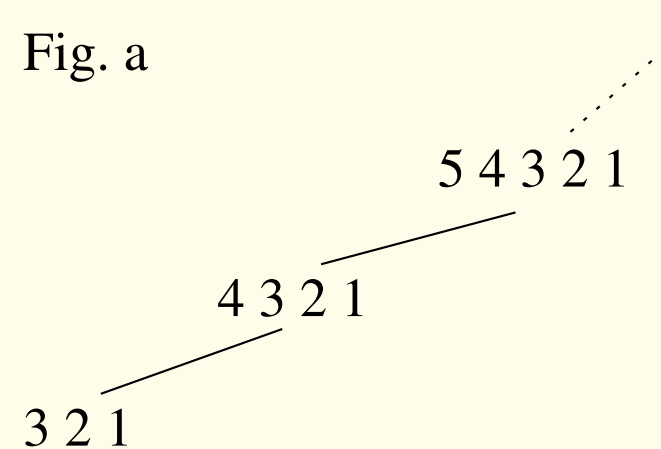
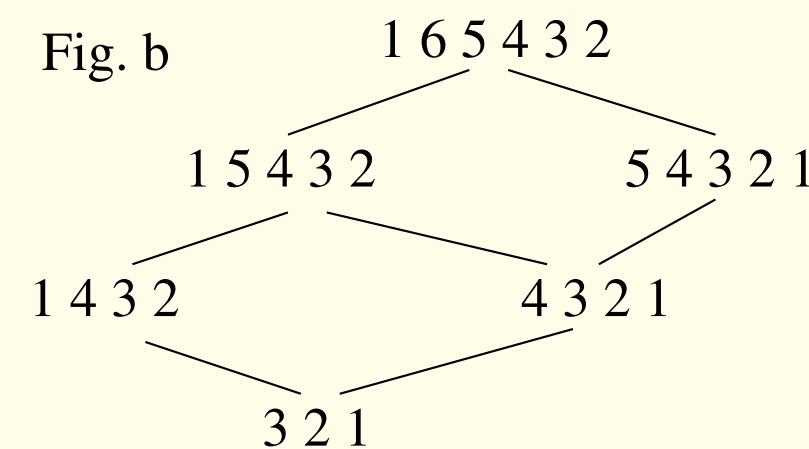
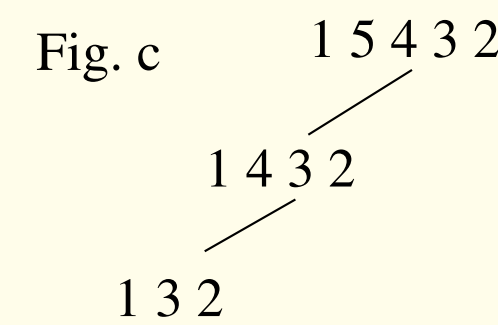
$$\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z)$$

or equivalently

$$\mu(x, y) = - \sum_{x < z \leq y} \mu(z, y)$$

Simple general cases

- if τ is a monotone permutation, then it covers only one element (which is in turn a monotone permutation);
- if $\tau = a_1 \cdots a_n$ is not monotone and such that either a_1 and a_2 are consecutive, or $\tau = 1n(n-1) \cdots 32$, or $\tau = n12 \cdots (n-2)(n-1)$, then τ covers precisely two elements;
- otherwise, τ covers precisely three elements.



- if τ is monotone, then $[\sigma, \tau]$ is a chain $\rightsquigarrow$ Möbius function easy to compute (Fig. a).
- if $\tau = a_1 \cdots a_n$ covers precisely two permutations and the first two entries of τ are not consecutive, then

- if σ is monotone (Fig. b), the interval $[\sigma, \tau]$ is the product of a chain of length 2 and a chain of length $n \rightsquigarrow$ Möbius function easy to compute:

$$\mu(\sigma, \tau) = \begin{cases} -1, & \text{if } \sigma \text{ has length } n-1; \\ 1, & \text{if } \sigma \text{ has length } n-2; \\ 0, & \text{otherwise.} \end{cases}$$

- if σ is not monotone (Fig. c), then necessarily each occurrence of σ in τ must involve a_1 . In this case, $[\sigma, \tau]$ is a finite chain, whose Möbius function is trivial to compute.

The case of one occurrence

Suppose $\sigma \leq \tau = a_1 - a_2 \cdots a_n$ and σ occurs in τ exactly once. We distinguish several cases, depending on whether the unique occurrence of σ in τ involves any of the elements a_1, a_2, a_n .

- If σ involves all of the three elements a_1, a_2, a_n , then $\sigma = \tau \rightsquigarrow \mu(\sigma, \tau) = 1$.
- If σ involves any two of the three elements a_1, a_2, a_n , then $[\sigma, \tau]$ is a chain $\rightsquigarrow$ Möbius function easy to compute.
- If only one among a_1, a_2, a_n is involved in the unique occurrence of σ in τ , there are three distinct cases to consider:

Proposition. Suppose that a_1 (a_2) occurs in σ (whereas a_2 (a_1) and a_n does not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2 (in which case $\mu(\sigma, \tau) = 1$).

Proposition. Suppose that a_n occurs in σ (whereas a_1 and a_2 does not). Then $\mu(\sigma, \tau) = 0$, unless $[\sigma, \tau]$ has rank 2. In this last case, if a_1 and a_2 are consecutive integers, then $\mu(\sigma, \tau) = 0$, otherwise $\mu(\sigma, \tau) = 1$.

- Last case:

Proposition. If σ does not involve a_1, a_2 and a_n , then $\mu(\sigma, \tau) = 0$.

Further work

- In the case of one occurrence the absolute value of μ is 0 or 1. In the general case (more than one occurrence) it is not true (e.g. $\mu(12, 2413) = 2$), actually it seems to be unbounded.
- Despite the closeness with the consecutive pattern poset, the computation of μ of an interval in the quasi-consecutive pattern poset can sometimes be very similar to the classical unrestricted case (pattern poset).
- The solution of the general case (more than one occurrence) may shed light on the differences and the analogies between the two extremal cases.