

Connections Between Rook Monoid Pattern Avoidance and Other Combinatorial Objects

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Outline

- 1 Definitions, Notation and Preliminary Results
- 2 A-reducibility
- 3 RC-Invariant Permutations
- 4 Q-residues

Definition of Rook Monoid

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Examples: $08170026 \in \mathcal{R}_8$, $\mathcal{R}_2 = \{00, 01, 02, 10, 20, 12, 21\}$

Rook Monoid Pattern Avoidance

Definition

Let $\rho \in \mathcal{R}_m$ and $\pi \in \mathcal{R}_n$. We say that π *contains* ρ as a pattern if there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\pi_{i_\ell} = 0$ if and only if $\rho_\ell = 0$ and for $\pi_{i_a}, \pi_{i_b} > 0$, $\pi_{i_a} > \pi_{i_b}$ if and only if $\rho_a > \rho_b$.

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Examples: 30012 contains 201, 2001, but not 102 or 20001.

Notation

Let Q be a set of rook patterns.

Define:

- $\mathcal{R}_n(Q) := \{\pi \in \mathcal{R}_n \mid \pi \text{ avoids } \rho \text{ for all } \rho \in Q\}$
- $r_n(Q) := |\mathcal{R}_n(Q)|$

Some Counting Results

Pattern q	$r_n(q)$	OEIS
1	1, 1, 1, 1, 1, ...	A000012
0	1, 2, 6, 24, 120, ...	A000142
01	2, 5, 16, 65, 326, ...	A000522
12	2, 6, 20, 70, 252, ...	A000984
00	2, 6, 24, 120, 720, ...	A000142
102	2, 7, 31, 159, 916, ...	A221958
012	2, 7, 31, 159, 921, ...	A221957
001	2, 7, 31, 165, 1031, ...	A193657
123	2, 7, 33, 183, 1118, ...	A086618
000	2, 7, 33, 192, 1320, ...	A006595

Coxeter Groups

Definition

Let I be an index set and $S = \{s_i \mid i \in I\}$.

For each pair (i, j) where $i, j \in I$, we associate $m(i, j) \in \mathbb{N} \cup \{\infty\}$ such that $m(i, j) = 1$ iff $i = j$ and $m(i, j) = m(j, i)$.

Let W be a the group with presentation $\langle S \mid (s_i s_j)^{m(i, j)} \rangle$, then (W, S) is called a *Coxeter system*.

Coxeter Groups of types A and B

- Coxeter group of type A (A_n)
 - n generators $s_0, s_1, \dots, s_{n-1}$
 - $m(i, i+1) = 3$
 - $m(i, j) = 2$ if $|i - j| > 1$
 - $A_n \cong S_{n+1}$

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 - $A_n \cong S_{n+1}$
- Coxeter group of type B (B_n)
 - n generators $s_0, s_1, \dots, s_{n-1}$
 - $m(0, 1) = 4$
 - $m(i, i+1) = 3, i \geq 1$
 - $m(i, j) = 2$ if $|i - j| > 1$

Coxeter Group of type B

Set of all “signed” permutation on $[n]$.

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Ex: $\overline{12345} \in B_5$, $\overline{2351746} \in B_7$

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Definition

If $w \in W$ and $w = s_{i_1} s_{i_2} \dots s_{i_l}$ is an expression of minimal length for w , then $i_1 i_2 \dots i_l$ is a *reduced expression* for w and l is the length of w , denoted $l(w)$.

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Example: 0101 is not reduced in A_n , but reduced in B_n .

Reduced Words

Definition

For any $w \in W$, define $\mathcal{R}(w)$ to be the set of reduced words of w .
If $S \subset W$, define $\mathcal{R}(S) = \bigcup_{w \in W} \mathcal{R}(w)$.

Example: $w = s_2 s_3 s_2 \in B_5$. $\mathcal{R}(w) = \{232, 323\}$.

A-Reduced Words

Definition (Stembridge, '97 [4])

$w \in B_n$ is *A-reduced* if $\mathcal{R}(w) \subset \mathcal{R}(A_n)$.

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Theorem (Stembridge, '97 [4])

For $w \in B_n$, the following are equivalent.

- 1 w is A-reduced.
- 2 Neither 0101 nor 1012101 occur as subwords of any $\mathbf{i} \in \mathcal{R}(w)$.
- 3 w avoids the patterns $\overline{12}$ and $\overline{132}$.

A-reducibility and rook monoids

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Theorem (D., Pudwell)

For all $n \geq 1$, the number of A-reduced elements of B_n is equal to $r_n(000)$.

Sketch of Bijection

We must provide a bijection ϕ from $\mathcal{R}_n(000)$ to the set of all A -reduced elements of B_n (those avoiding $\overline{12}$ and $\overline{13\overline{2}}$).

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Step 2: If $\pi \in \mathcal{R}_n(000)$ contains exactly one zero, then $\pi = \pi_1 \dots \pi_{i-1} 0 \pi_{i+1} \dots \pi_n$ where $a \in [n]$ does not appear in π . Define $\phi(\pi) := \pi_1 \pi_2 \dots \pi_{i-1} \bar{a} \pi_{i+1} \dots \pi_n$.

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What happens if π contains two zeros?

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$\phi(635108902) = 6\bar{7}5\bar{4}\bar{3}89\bar{1}\bar{2}$. Avoids $\bar{1}\bar{2}$ and $1\bar{3}\bar{2}$.

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Step 3: Remove the first two elements in the list and start replacing barred elements from left to right starting with the third element in the list from step 2.

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635108902

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$$r_n(321) = \sum_{k=0}^n \binom{n}{k}^2 C_k$$

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$$r_n(321) = \sum_{k=0}^n \binom{n}{k}^2 C_k$$

Theorem (Egge, 2010, [3])

$$|\mathcal{S}_{2n}^{rc}(4321)| = \sum_{k=0}^n \binom{n}{k}^2 C_k$$

Bijection

Egge's bijection: Match (P', Q', π_o, π_e) to a member of $S_{2n}^{rc}(4321)$.

- $P' \subseteq [n]$
- $Q' \subseteq [n]$
- $|P'| = |Q'| = k, 0 \leq k \leq n$
- $\pi_o \in S_k(321)$
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Our addition: Given $\pi = 01027803 \in \mathcal{R}_8(321)$.

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- $|P'| = |Q'| = k$, $0 \leq k \leq n$ $\mathbf{k} = 5$
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- $\pi_o \in S_k(321)$ $\pi_o = \mathbf{12783}$
- $\pi_e \in S_{n-k}(21)$ $\pi_e = \mathbf{123}$

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First, fix an infinite sequence $Q = \{q_0(x), q_1(x), q_2(x), \dots\}$ of polynomials where the degree of q_k is k . Let $p(x)$ be a polynomial.

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Q-residues

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Note: $D^n(p)$ is constant and is called the Q-residue of p .

Example of a Q-residue

Define

$$q_0(x) = 1$$

$$q_1(x) = 2x + 3$$

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A Strange Sequence

The sequence

1, 2, 7, 31, 165, 1031, 7423, 60621, 554249, 5611771, ... is
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WHY??

A Last Plea

If you know anything about Q-residues or have some insight we have not thought of, please let us know!

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Thank you!!

References

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