q-Analogues of Convolutions of Fibonacci Numbers

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The pattern μ

- Avgustinovich, Kitaev, Valyuzhenich: avoidance in permutations
- Jones, Kitaev, Remmel: distribution in cycles of permutations
- LoBue, Remmel: distribution in words

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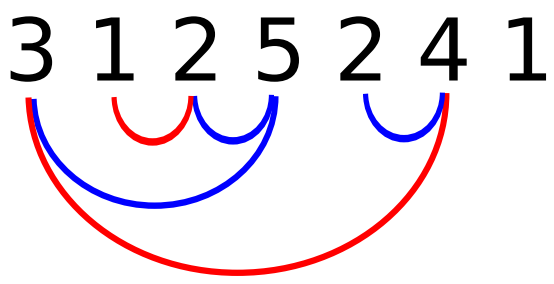
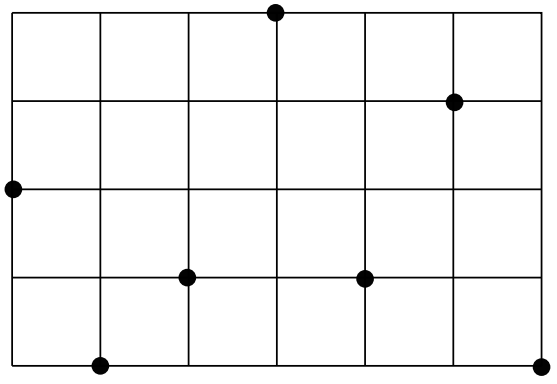
$$w = w_1 w_2 \dots w_n \in NR([k]), w_i \neq w_{i+1}$$

Definition

The pair $\langle w_i, w_j \rangle$ is a *μ-match* in w if $i < j$, $w_i < w_j$, and there is no $i < l < j$ such that $w_i \leq w_l \leq w_j$.





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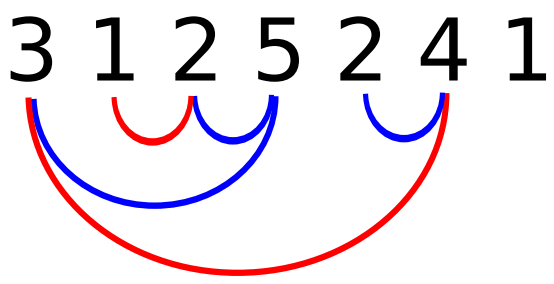

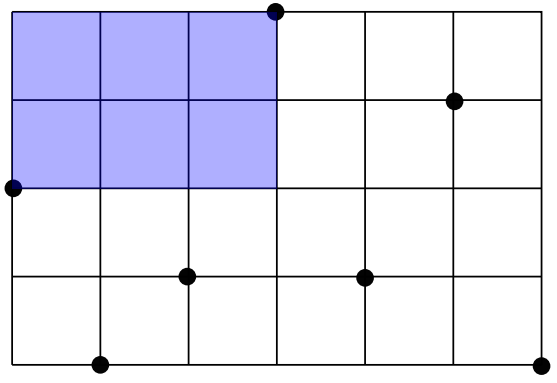

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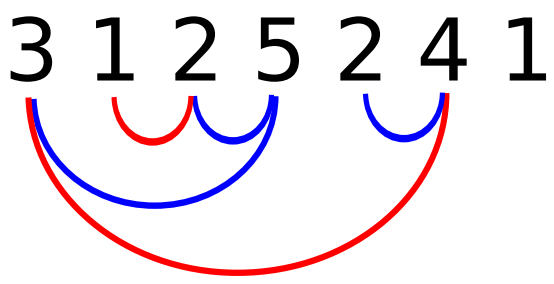

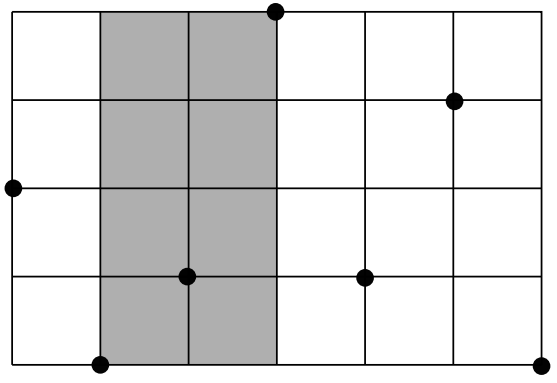

$$N^{i,j}(p,q,t) = \sum_{n \geq 1} N_n^{i,j}(p,q)t^n, \text{ where}$$

$$N_n^{i,j}(p,q) = \sum_{\substack{w \in NR([3]) \\ |w|=n \\ w_1=i, w_n=j}} p^{nontriv(w)}q^{triv(w)}$$

A refinement of the generating function for all words in $NR([3])$:

$$N(p,q,t) = 1 + \sum_{i,j \in [3]} N^{i,j}(p,q,t)$$

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A refinement of the generating function for all words in $NR([3])$:

$$N(p,q,t) = 1 + \sum_{i,j \in [3]} N^{i,j}(p,q,t)$$

Special simplification in the case $k = 3$

The only possible matches are $\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,3 \rangle$. All nontrivial matches are occurences of the consecutive pattern 13.

Computing $N(p, q, t)$ $p^{nontriv(w)}q^{triv(w)}t^{|w|}$

- Each word $w \in NR([3])$ belongs to a state of a finite automaton.
- The state records which μ matches may occur when a letter is appended to w.

Example: $w = 2$

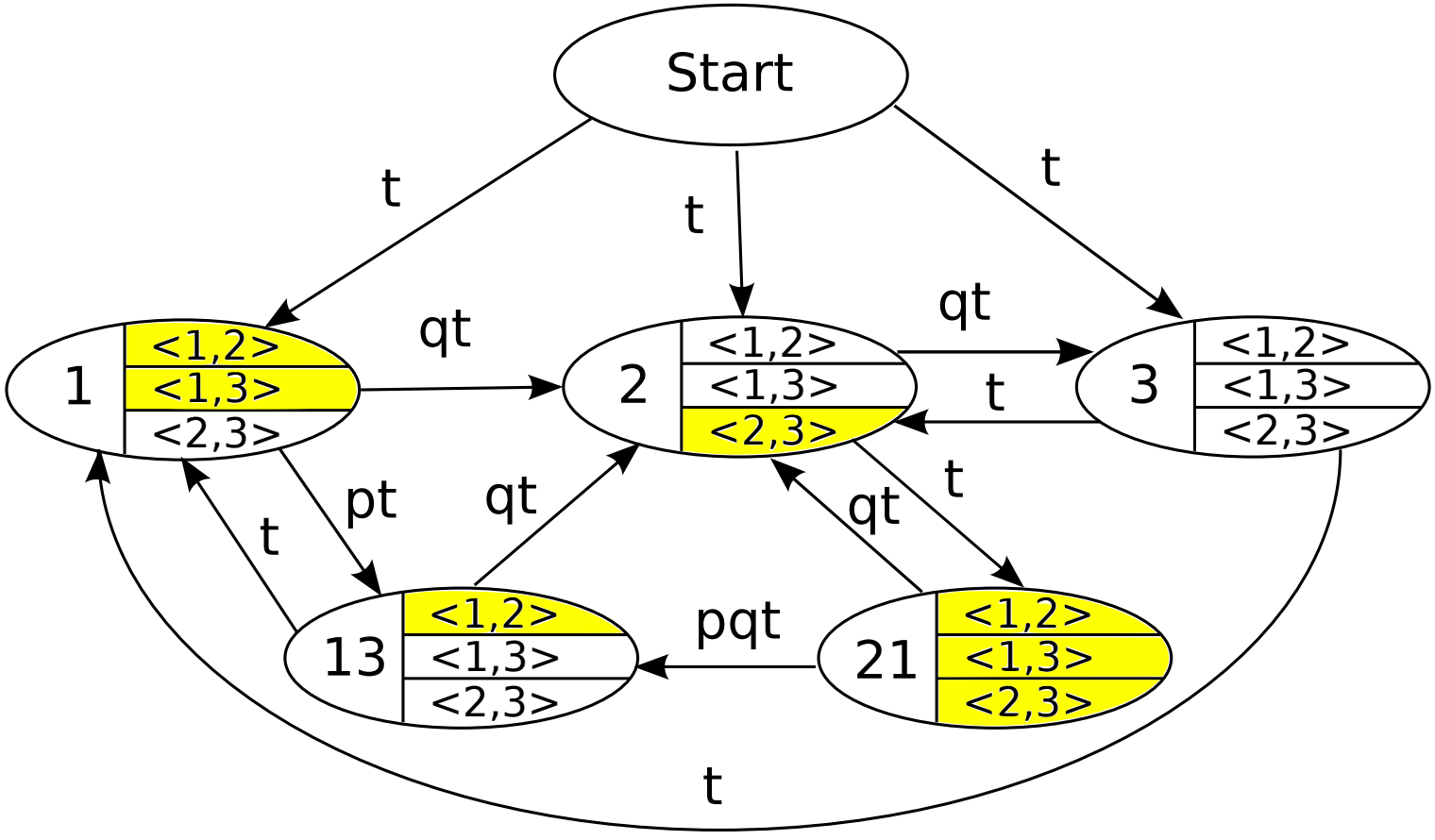
$\langle 1, 2\rangle, \langle 1, 3\rangle$ matches are not possible
$\langle 2, 3\rangle$ match is possible

Also in this state is the word $w = 12$.

- Transition from state i to state j if you can append a letter to a word in state i to obtain a word in state j.
- Weight the edges with the change in length (t) and any μ-matches completed by appending the last letter (p's and q's).

The automaton for $k = 3$

$p^{nontriv(w)} q^{triv(w)} t^{|w|}$



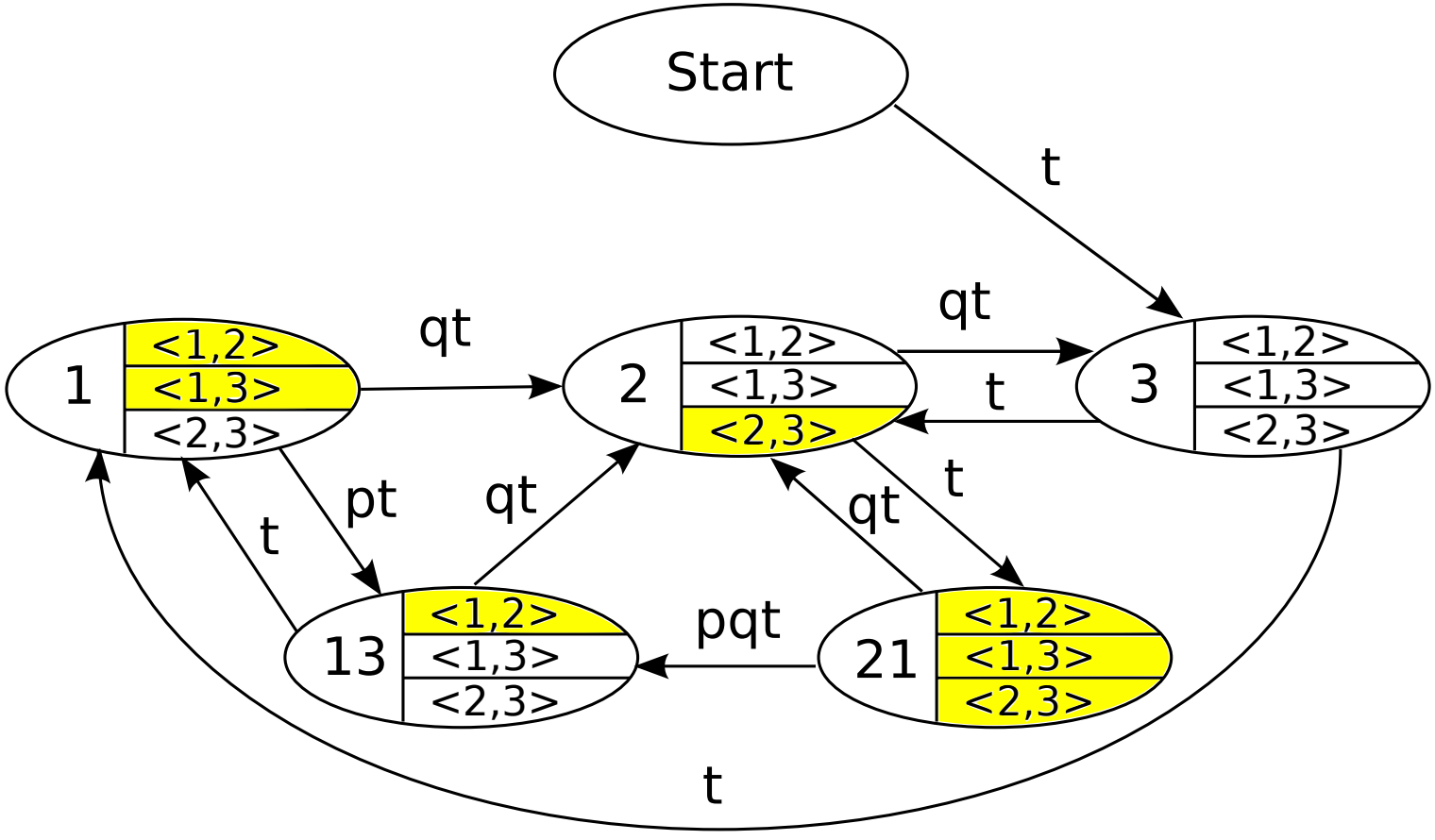
This gives rise to a system of a equations, one for each state.

$$S_3 = t + qtS_2$$

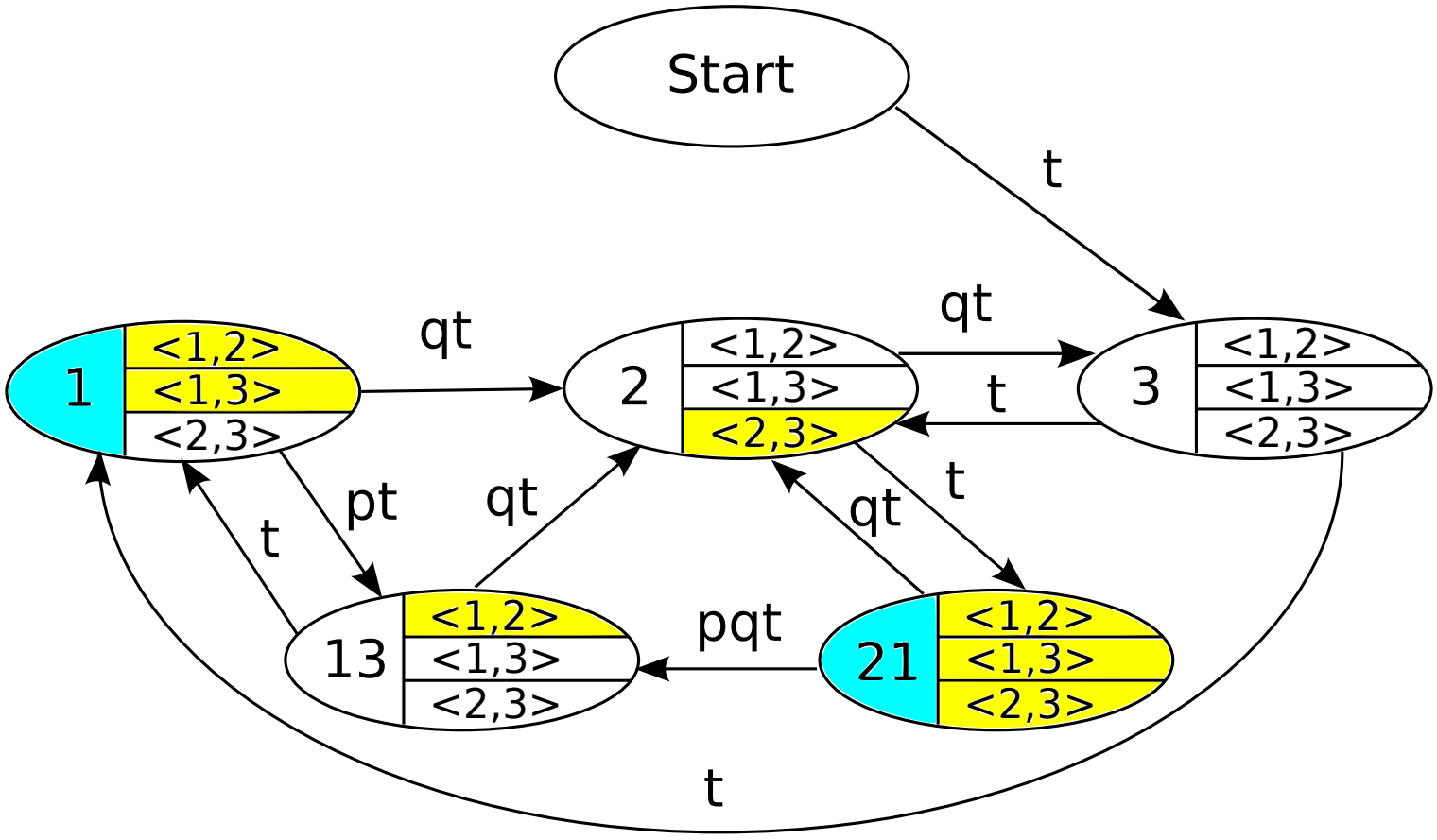
Solve the system for all five states and sum them to get $N(p, q, t)$.

Computing $N^{3,1}(p, q, t)$

$p^{nontriv(w)} q^{triv(w)} t^{|w|}$



This system of equations gives only the words starting with 3.



This system of equations gives only the words starting with 3.

Important property

All words in a given state have the same last letter.

To get words ending in 1, only sum the shaded states.

$$N^{3,1}(p,q,t) = \frac{t^2 + t^3 - p(1-q)^2 t^5}{1 - 2qt^2 - q^2 t^3 - pt^2(1 + q^2 t + 2q(q-1)t^2)}$$

$$N^{3,1}(p,q,t) = \frac{t^2 + t^3 - p(1-q)^2t^5}{1 - 2qt^2 - q^2t^3 - pt^2(1 + q^2t + 2q(q-1)t^2)}$$

Theorem

For $n \geq 2$, $N_n^{3,1}(0,1) = F_{n-2}$.

Proof.

$$N^{3,1}(0,1,t) = \frac{t^2(1+t)}{1 - 2qt^2 - q^2t^3}$$
$$= \frac{t^2}{1 - t - t^2}$$

□

Theorem

For $n \geq 2$, $N_n^{3,1}(0,1) = F_{n-2}$.

Proof.

Let $\mathcal{C}_n = \{w \in NR([3]) : |w| = n, w_1 = 3, w_n = 1, w \text{ has no 13s}\}$.
Then $N_n^{3,1}(0,1) = |\mathcal{C}_n|$.

$\mathcal{C}_2 = \{31\} \to F_0$
$\mathcal{C}_3 = \{321\} \to F_1$

For $n \geq 4$,

$$\left.\begin{array}{l} 3\ldots121 \to \mathbf{3}\ldots\mathbf{1}21 \\ 3\ldots321 \to \mathbf{3}\ldots\mathbf{3}21 \\ 3\ldots231 \to \mathbf{3}\ldots\mathbf{2}31 \end{array}\right\} \to F_{n-3}$$

$$\left.\begin{array}{l} 3\ldots121 \to \mathbf{3}\ldots\mathbf{1}21 \end{array}\right\} \to F_{n-4}$$

□

Theorem

For all $r \geq 0$,

$$N^{3,1}(p,1,t)|_{p^r} = \frac{t^{2r+2}}{(1-t-t^2)^{r+1}}.$$

so that for all $n, r \geq 0$, $N_n^{3,1}(p,1)|_{p^r} = \sum_{\substack{j_1,\ldots,j_{k+1}\geq 0 \\ n-2r-2=\sum_{i=1}^{r+1} j_i}} F_{j_1}\cdots F_{j_{r+1}}.$

Proof.

Unique factorization of words with r 13s produces $r+1$ words with no 13s:

$$\boxed{3\ldots1}\boxed{3\ldots1}\boxed{3\ldots1}\boxed{3\ldots1}\boxed{3\ldots1}$$

$$N^{3,1}(p,1,t)|_{p^r} = \prod_{i=1}^{r+1} N^{3,1}(0,1,t) = \frac{t^{2r+2}}{(1-t-t^2)^{r+1}}.$$

It follows that

① $N_n^{3,1}(0,q)$ is a q-analogue of F_{n-2}, and

② $N_n^{3,1}(p,q)|_{p^r}$ is a q-analogue of $\sum_{\substack{j_1,\ldots,j_{r+1}\geq 0 \\ n-2r-2=\sum_{i=1}^{r+1} j_i}} F_{j_1}\cdots F_{j_{r+1}}$.

n	$N_n^{3,1}(0,q)$	n	$N_n^{3,1}(0,q)$
0	0	8	$12q^3+q^4$
1	0	9	$8q^3+13q^4$
2	1	10	$28q^4+6q^5$
3	1	11	$16q^4+38q^5+q^6$
4	$2q$	12	$64q^5+25q^6$
5	$2q+q^2$	13	$32q^5+104q^6+8q^7$
6	$5q^2$	14	$144q^6+88q^7+q^8$
7	$4q^2+4q^3$	15	$64q^6+272q^7+41q^8$

$$N^{3,1}(0,q,t) = \frac{t^2+t^3}{1-2qt^2-q^2t^3}$$

The denominator gives a recursion that we can also prove combinatorially.

Theorem

For $n \geq 4$, $N_n^{3,1}(0,q) = 2qN_{n-2}^{3,1}(0,q) + q^2N_{n-3}^{3,1}(0,q)$

Proof.

$$\left.\begin{array}{l} 3\ldots121 \to \mathbf{3}\ldots\mathbf{1}21 \\ 3\ldots2321 \to \mathbf{3}\ldots\mathbf{23}\mathbf{1} \\ 3\ldots3231 \to \mathbf{3}\ldots\mathbf{32}\mathbf{1} \\ 3\ldots1231 \to \mathbf{3}\ldots\mathbf{1}231 \end{array}\right\} \to qN_{n-2}^{3,1} + q^2N_{n-3}^{3,1}$$

□

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With these initial values and the recursion, we can show

$$N_{2n+1}^{3,1}(0,q) = 2^{n-1}q^{n-1} + \text{ higher order terms.}$$

n	$N_n^{3,1}(0,q)$	n	$N_n^{3,1}(0,q)$
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With these initial values and the recursion, we can show

$$N_{2n}^{3,1}(0,q) = (n+2)2^{n-3}q^{n-1} + \text{ higher order terms.}$$

n	$N_n^{3,1}(0,q)$	n	$N_n^{3,1}(0,q)$
0	0	8	$12q^3+q^4$
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$N_{3n}^{3,1}(0,q) = (1+2n(n-1))q^{2n-2} +$ lower order terms

$N_{3n+1}^{3,1}(0,q) = (2n)q^{2n-1} +$ lower order terms

$N_{3n+2}^{3,1}(0,q) = q^{2n} +$ lower order terms

n	$N_n^{3,1}(0,q)$	n	$N_n^{3,1}(0,q)$
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$$N_{2n+3}^{3,1}(0,q)|_{q^{n+k}} = \begin{cases} 0 & \text{if } n < 3k-2, \\ 1 & \text{if } n = 3k-2, \\ 4k & \text{if } n = 3k-1, \\ 2^j\left(4\binom{2k+j+1}{2k-1}+\binom{2k+j}{2k}\right) & \text{if } n = 3k+j, \quad j \geq 0. \end{cases}$$

$$N^{3,1}(p,q,t) = \frac{t^2+t^3-p(1-q)^2t^5}{1-2qt^2-q^2t^3-pt^2(1+q^2t+2q(q-1)t^2)}$$

$$= \frac{t^2+t^3-p(1-q)^2t^5}{1-2qt^2-q^2t^3}\,\frac{1}{1-p\left(\frac{t^2(1+q^2t+2q(q-1)t^2)}{1-2qt^2-q^2t^3}\right)}$$

$$= \frac{t^2(1+t)}{1-2qt^2-q^2t^3}+$$

$$\sum_{r\geq 1} p^r \boxed{\frac{t^{2r+2}(1+tq+q(q-1)t^2)^2(1+q^2t+2q(q-1)t^2)^{r-1}}{(1-2qt^2-q^2t^3)^{r+1}}}$$

$$N^{3,1}(p,q,t)|_p = \frac{t^4(1+tq+q(q-1)t^2)^2}{(1-2qt^2-q^2t^3)^2}$$

In this case, $N_n^{3,1}(p,q)|_p$ is a q-analogue of $\sum_{j=0}^{n-4} F_j F_{n-4-j}$.

n	$N_n^{3,1}(p,q)\|_p$	n	$N_n^{3,1}(p,q)\|_p$
0	0	9	$24q^3+14q^4$
1	0	10	$12q^3+51q^4+8q^5$
2	0	11	$66q^4+62q^5+2q^6$
3	0	12	$28q^4+162q^5+45q^6$
4	1	13	$172q^5+230q^6+18q^7$
5	$2q$	14	$64q^5+475q^6+202q^7+3q^8$
6	$2q+3q^2$	15	$432q^6+768q^7+108q^8$
7	$8q^2+2q^3$	16	$144q^6+1320q^7+789q^8+32q^9$
8	$5q^2+14q^3+q^4$	17	$1056q^7+2388q^7+522q^9+4q^{10}$

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$N_{2n+4}^{3,1}(p,q)|_p = (n+3)2^{n-2}q^n +$ higher order terms

$N_{2n+5}^{3,1}(p,q)|_p = (18+13n+n^2)2^{n-3}q^{n+1} +$ higher order terms

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$N_{3n+3}^{3,1}(p,q)|_p = n(2n^2-2n+3)q^{2n} +$ lower order terms

$N_{3n+4}^{3,1}(p,q)|_p = 2n^2q^{2n+1} +$ lower order terms

$N_{3n+5}^{3,1}(p,q)|_p = nq^{2n+2} +$ lower order terms

Theorem

$$N_n^{1,1}(0,1) = N_n^{3,3}(0,1) = F_{n-3}$$
$$N_n^{2,1}(0,1) = N_n^{3,2}(0,1) = F_{n-2}$$
$$N_n^{1,2}(0,1) = N_n^{2,3}(0,1) = F_{n-3} + (-1)^n$$
$$N_n^{1,3}(0,1) = F_{n-4} + (-1)^{n-1}$$
$$N_n^{2,2}(0,1) = F_{n-2} + (-1)^{n-1}$$

$$N^{1,1}(0,q,t) = \frac{t - 2qt^3 - q^2t^4}{1 - 3qt^2 - 2q^2t^3 - (q^3 - q^2)t^4}$$

n	$N_n^{1,1}(0,q)$	$N_n^{1,1}(0,1)$	n	$N_n^{1,1}(0,q)$	$N_n^{1,1}(0,1)$
0	0	0	6	$5q^3$	5
1	1	1	7	$5q^3 + 6q^4$	11
2	0	0	8	$18q^4 + 3q^5$	21
3	q	1	9	$13q^4 + 29q^5 + q^6$	43
4	q^2	1	10	$59q^5 + 26q^6$	85
5	$2q^2 + q^3$	3	11	$34q^5 + 122q^6 + 15q^7$	171

Jacobsthal numbers: $J_n = J_{n-1} + 2J_{n-2}$

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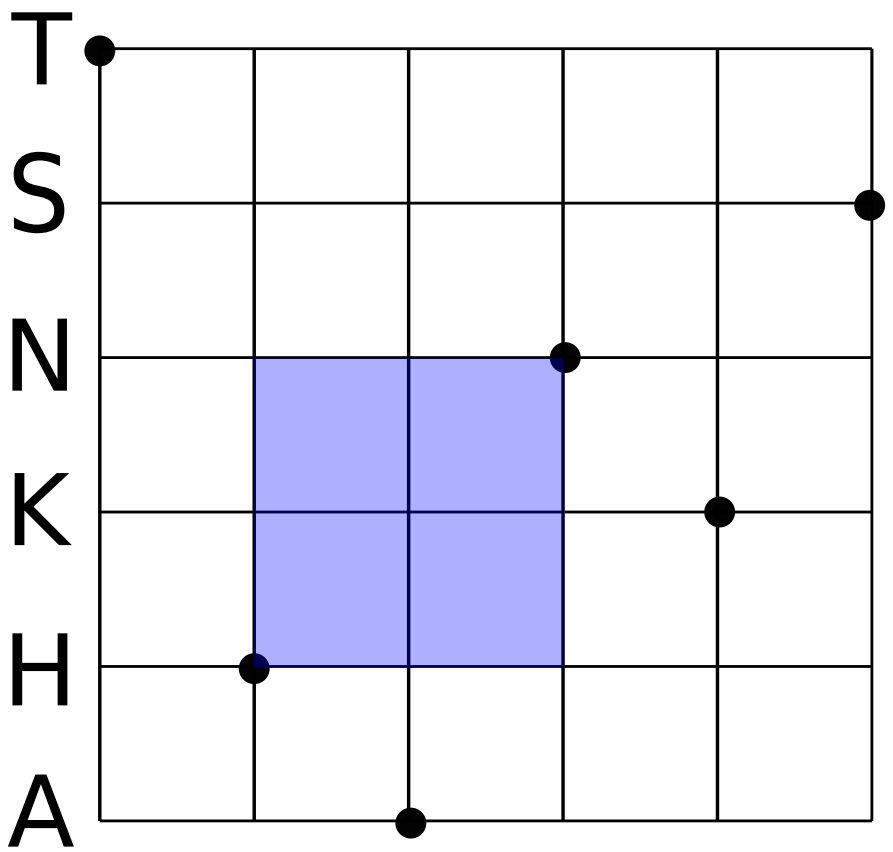
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