Word-Representable Graphs and Permutation Patterns

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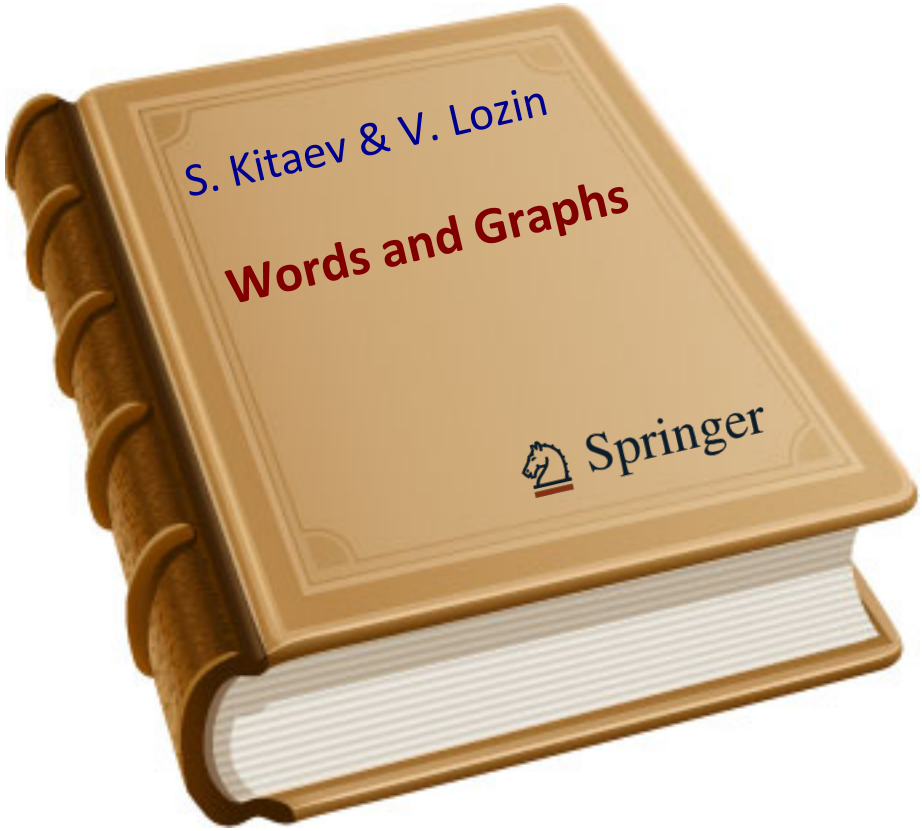
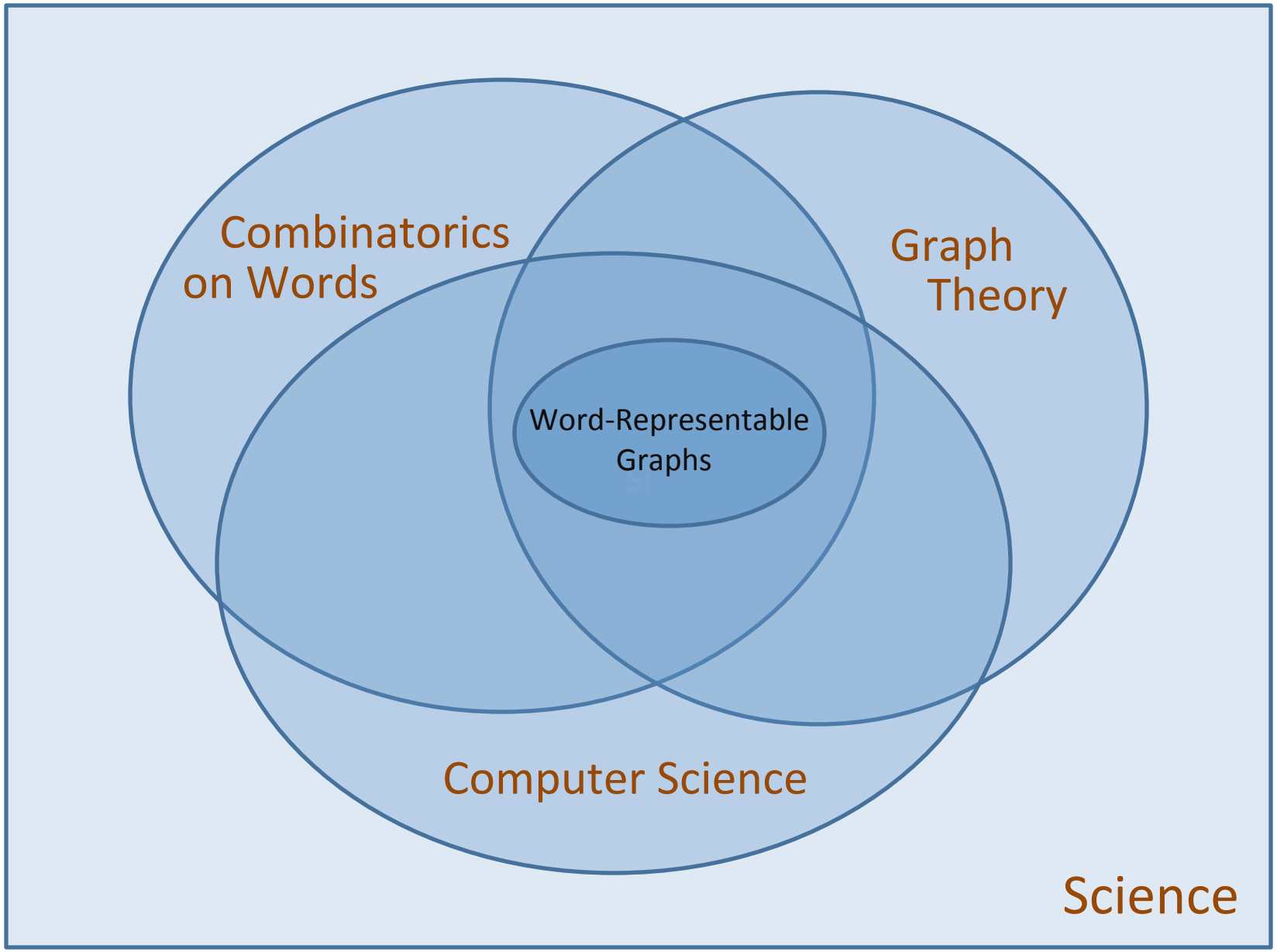
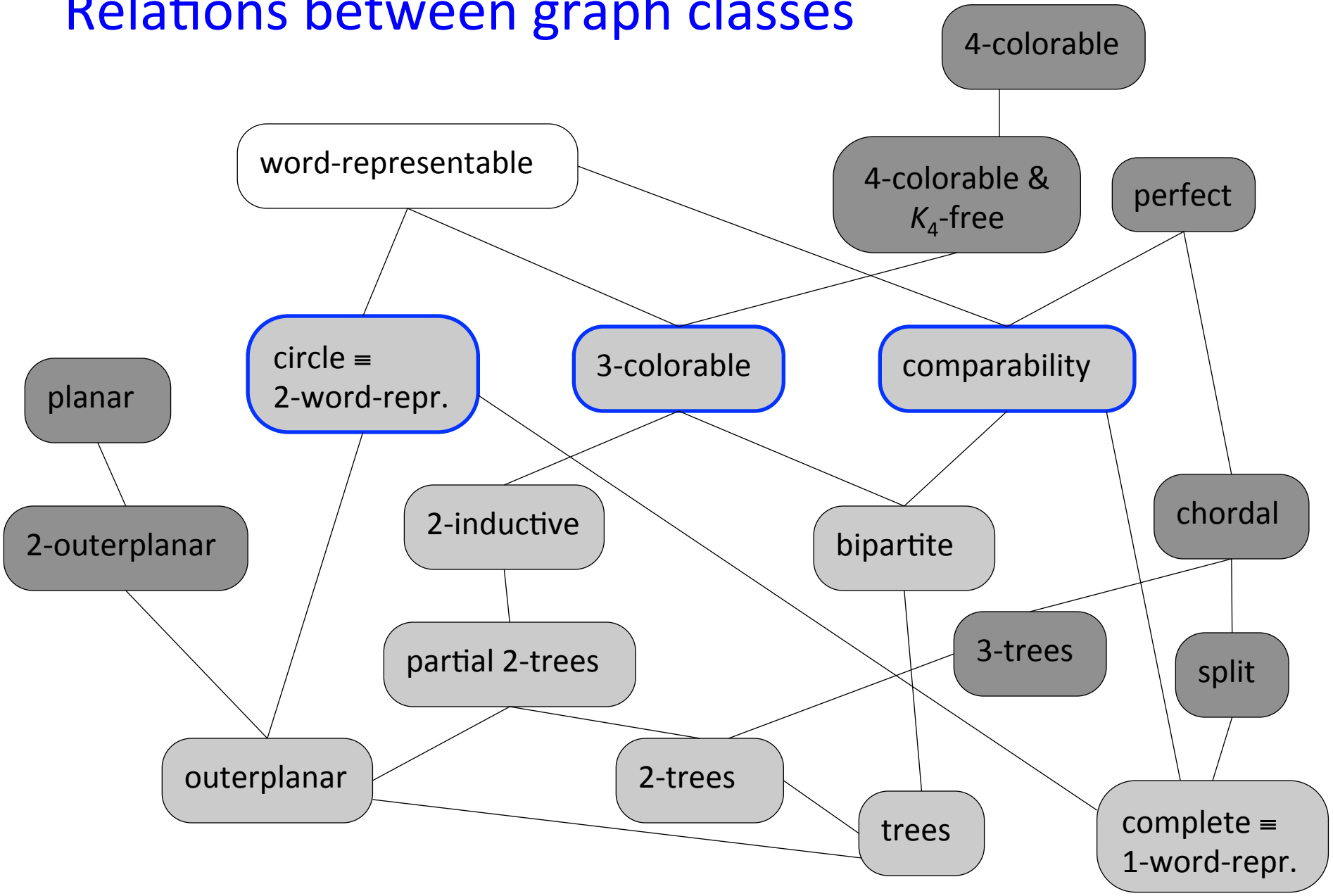

S. Kitaev & V. Lozin
Words and Graphs
Springer


Combinatorics
on Words
Graph
Theory
Word-Representable
Graphs
Computer Science
Science

Relations between graph classes

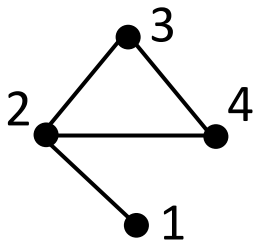


Motivation for Research

- Study of the Perkins semigroup (original motivation) – **Algebra**
- Scheduling robots on a path, periodically – **Computer Science**
- Generalization of well-known classes of graphs – **Graph Theory**
- Modification of graphs studied before – **Combinatorics on Words**
- Beautiful mathematics – **Mathematics**
- Just fun – **Human Sciences**

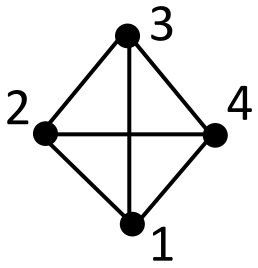
Definitions by examples

In the word 2 3 1 2 1 3, letters 2 and 3 alternate, 1 and 2 alternate,
1 and 3 do **not** alternate

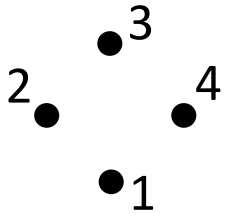


← A simple graph (no loops, no multiple edges); this is an unlabelled graph (renaming vertices gives the same graph)

The graph above is word-representable (or, representable) because the word 1 2 1 3 4 2 3 represents it, that is, (*a*,*b*) is an edge in the graph **if and only if** *a* and *b* alternate in the word.



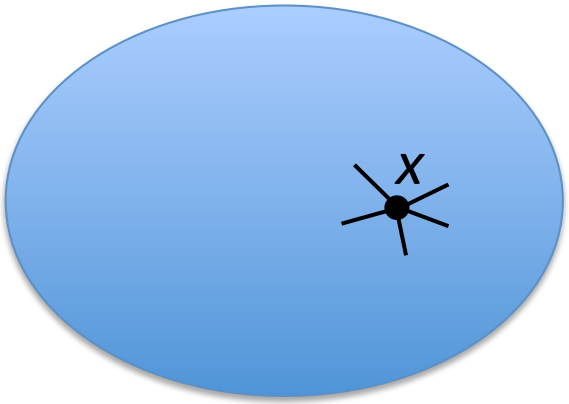
is representable by 1 2 3 4 or by 1 2 3 4 1 2 3 4



is representable by 1 2 3 4 4 3 2 1

Hereditary property

a word-representable graph

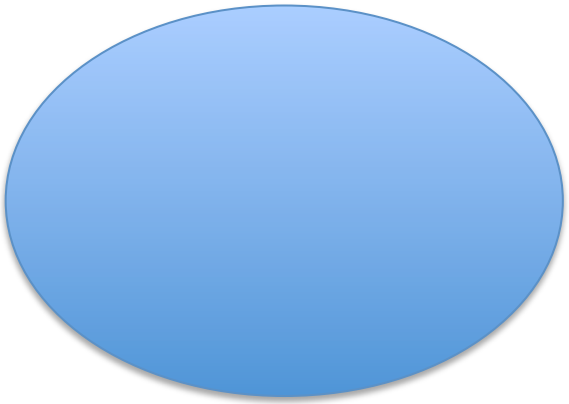


its word-representant

… *x* … *x* … *x* … *x* …

Hereditary property

a word-representable graph



its word-representant

… … … … …

k-representability and graph's representation number

2 4 3 3 2 1 4 4 2 3 1 1 is a 3-uniform word (each letter occurs three times)

2 3 1 5 4 is a 1-uniform word or permutation

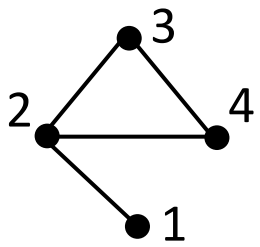
Observation: A *k*-uniform word and **any** of its cyclic shifts represent the same graph.

A graph is *k*-word-representable if there exists a *k*-uniform word representing it.

Theorem (SK & Pyatkin) A graph is representable **iff** it is *k*-representable for some *k*

Theorem (SK & Pyatkin) *k*-representability implies (*k*+1)-representability

Graph's representation number is the least *k* such that the graph is *k*-representable. For non-word-representable graphs $k=\infty$.



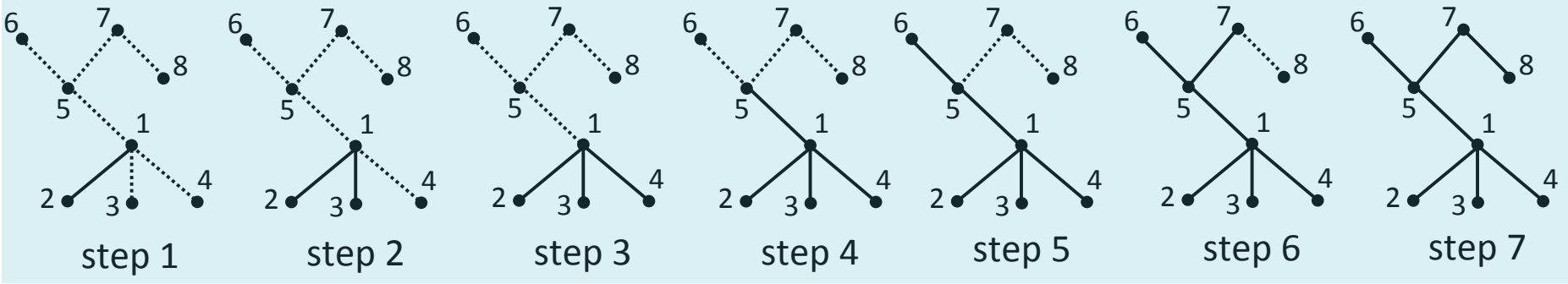
← This graph's representation number is 2 since it **can** be represented by the word 1 2 1 3 4 2 3 4 but **cannot** be represented by a word having one copy of each letter.

Cliques = 1-word-representable graphs

Questions to ask

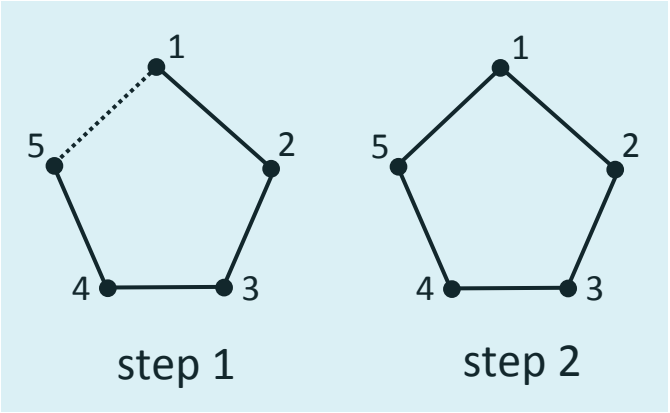
- Are all graphs word-representable?
- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
- What is graph's representation number? (minimal length of a word-representant)
- How hard is it to decide whether a graph is representable or not? (complexity)
- Which graph operations preserve (non-)word-representability?

Trees are 2-word-representable



Step 1	$w_1 = 1212$
Step 2	$w_2 = 123132$
Step 3	$w_3 = 12341432$
Step 4	$w_4 = 5152341432$
Step 5	$w_5 = 656152341432$
Step 6	$w_6 = 65617572341432$
Step 7 (the tree itself)	$w_7 = 6561758782341432$

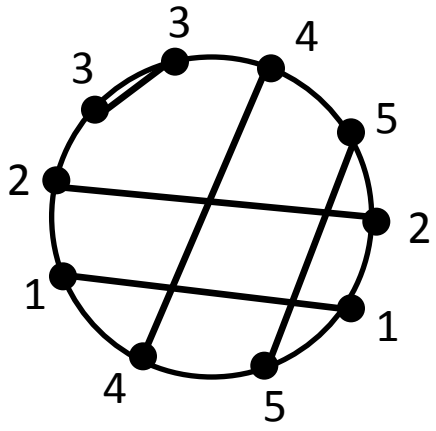
Cycle graphs are 2-word-representable

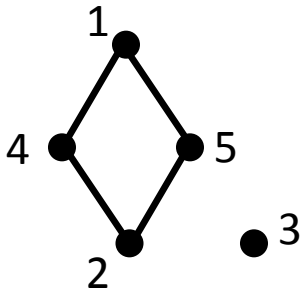


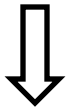
Step 1: Represent the path 1-2-3-4-5 using the approach to represent trees as follows: 1212 → 121323 → 12132434 → 1213243545.

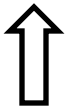
Step 2: Take the 1-letter cyclic shift 5121324354 and swap the first two letters to obtain the desired representation of the cycle 1521324354.

Circle graphs = 2-word-representable graphs









Reading the letters clockwise:

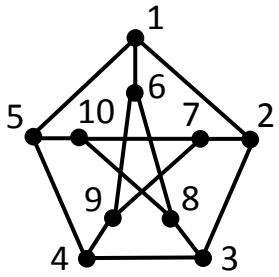
1 2 3 3 4 5 2 1 5 4

This is a 2-representation of the graph

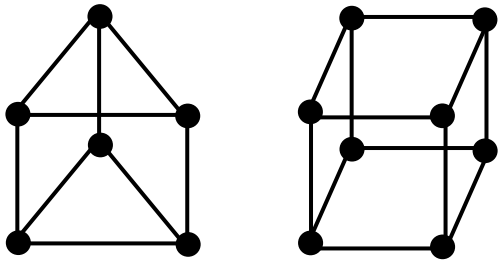
Theorem (Halldórsson, SK, Pyatkin) 2-representable graphs = circle graphs

3-word-representable graphs

Classification is unknown! All we know about these graphs are as follows:

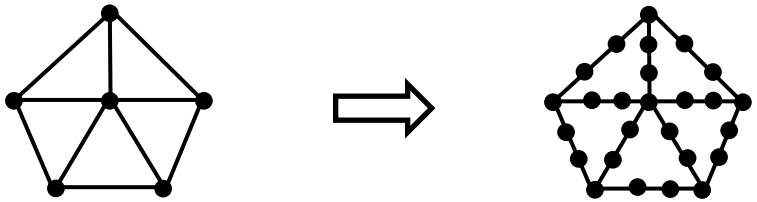


The Petersen graph being non-2-word-representable can be represented by
1 3 8 7 2 9 6 10 7 4 9 3 5 4 1 2 8 3 10 7 6 8 5 10 1 9 4 5 6 2



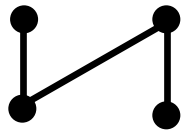
Theorem (SK & Pyatkin) Every prism is 3-representable

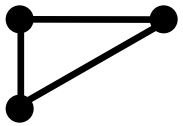
Theorem (SK & Pyatkin) 3-subdivision of **any** graph is 3-word-representable

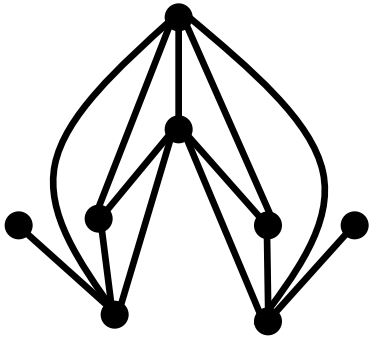


Comparability graphs

Edges can be oriented to form a transitive orientation

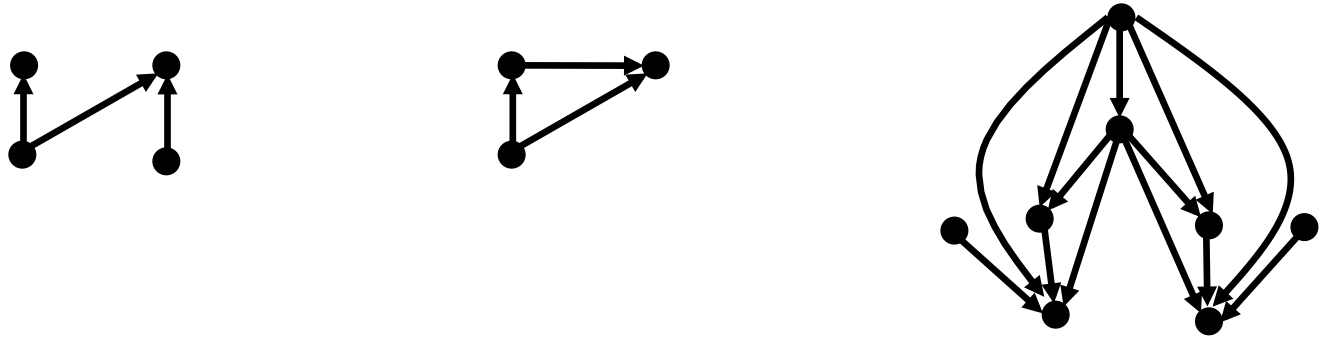




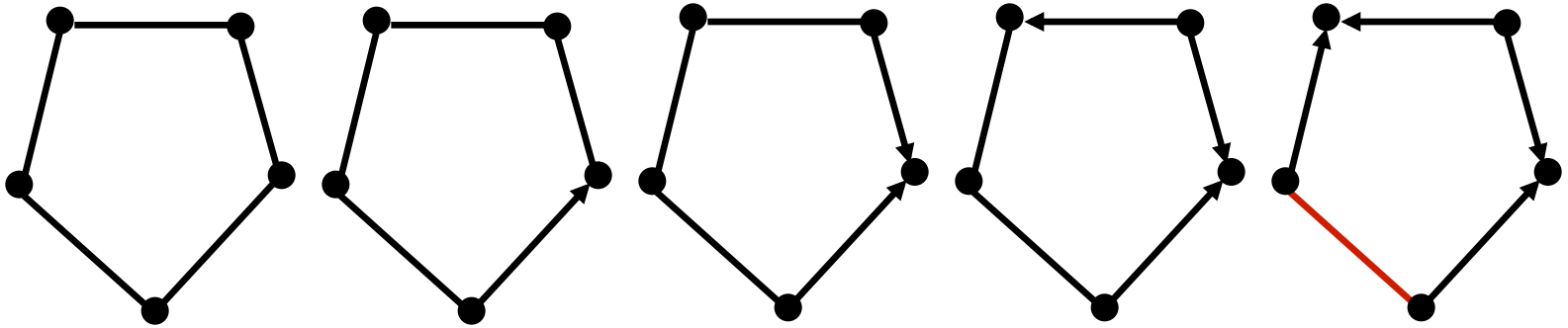


Comparability graphs

Edges can be oriented to form a transitive orientation

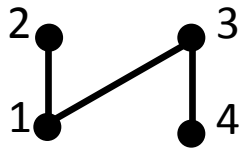


Comparability graphs correspond to partial orders



not a comparability graph

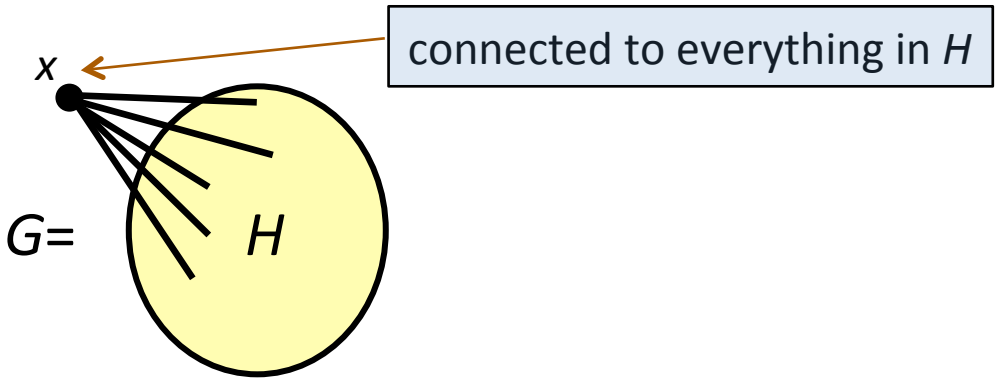
Permutationally representable graphs



is permutationally representable by 1243 1432 4123
(note that this is **not** the shortest possible representation)

Theorem (SK & Seif) Comparability graphs = permutationally-representable graphs

Significance of permutational representability



Theorem (SK & Pyatkin) *G* is representable **iff** *H* is permutationally representable

Theorem (SK & Pyatkin) *G* is representable ⇨ the neighbourhood of each node is permutationally representable (is a comparability graph)

Maximal Clique is polynomially solvable on word-representable graphs

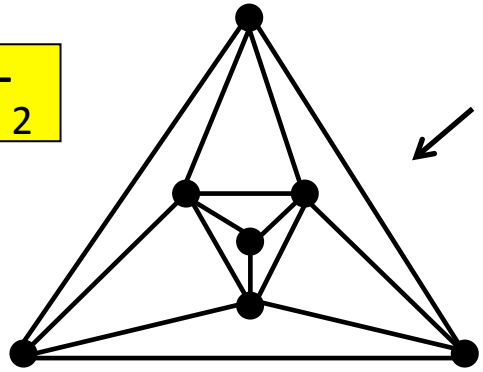
- A clique is contained within some neighborhood
- Neighborhoods induce comparability graphs
- Maximal Clique is polynomially solvable on comparability graphs

Significance of permutational representability

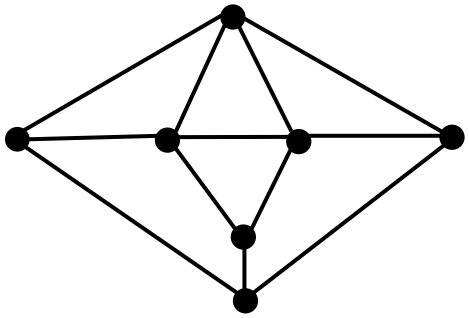
Converse to the last theorem does **not** hold:

Theorem (Halldórsson, SK, Pyatkin) *G* is representable $\not\Leftarrow$ the neighbourhood of each node is permutationally representable (is a comparability graph)

co-T_2

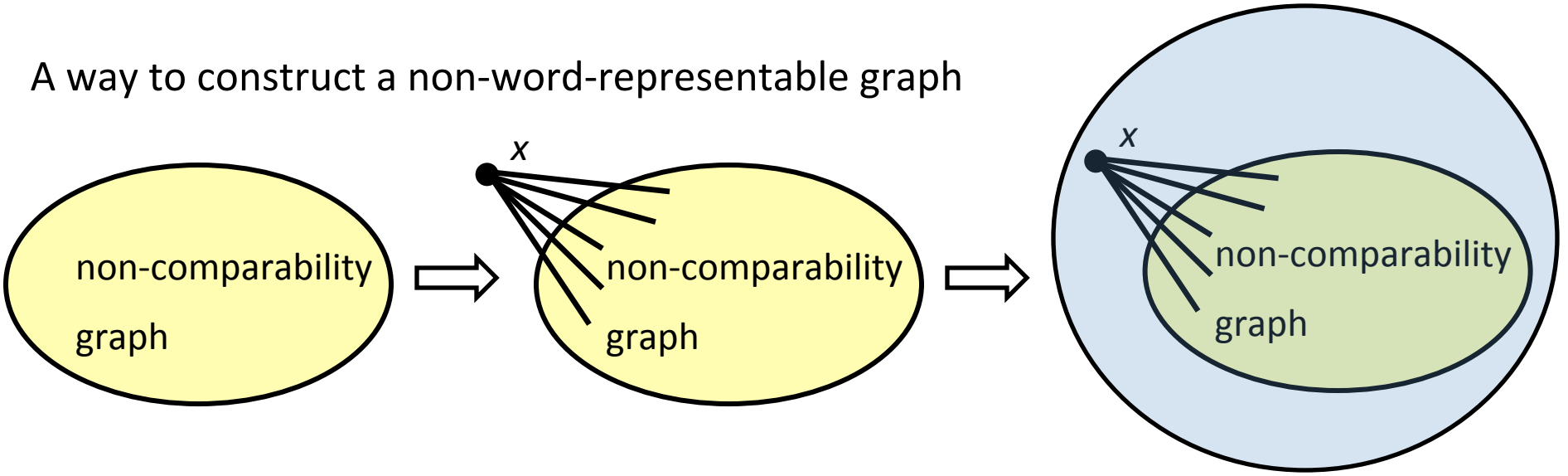


non-word-representable graphs

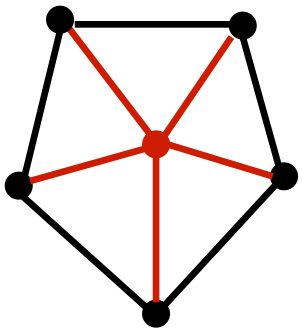


Non-word-representable graphs

A way to construct a non-word-representable graph



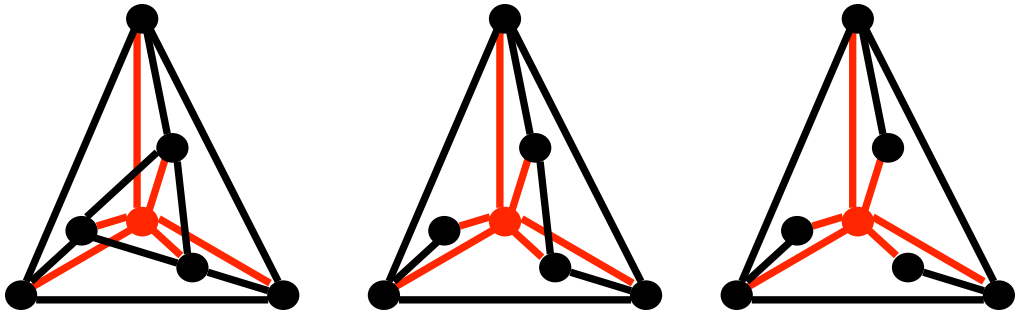
The smallest non-word-representable graph (the only one on 6 nodes)



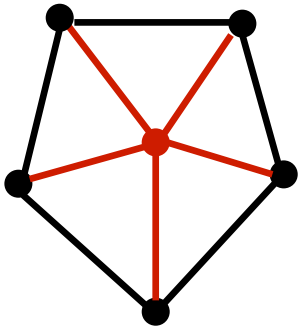
Wheel W_5

All odd wheels on more than 5 nodes are non-word-representable graphs

Other small non-representable graphs

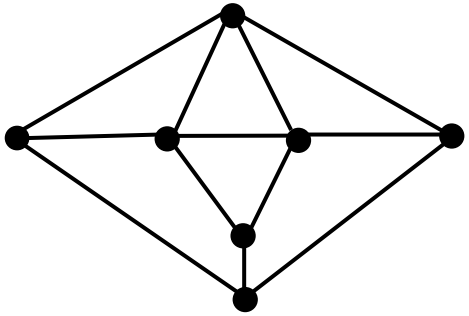


Non-word-representable graphs



non-word-representable ⇨

almost all graphs are non-word-representable

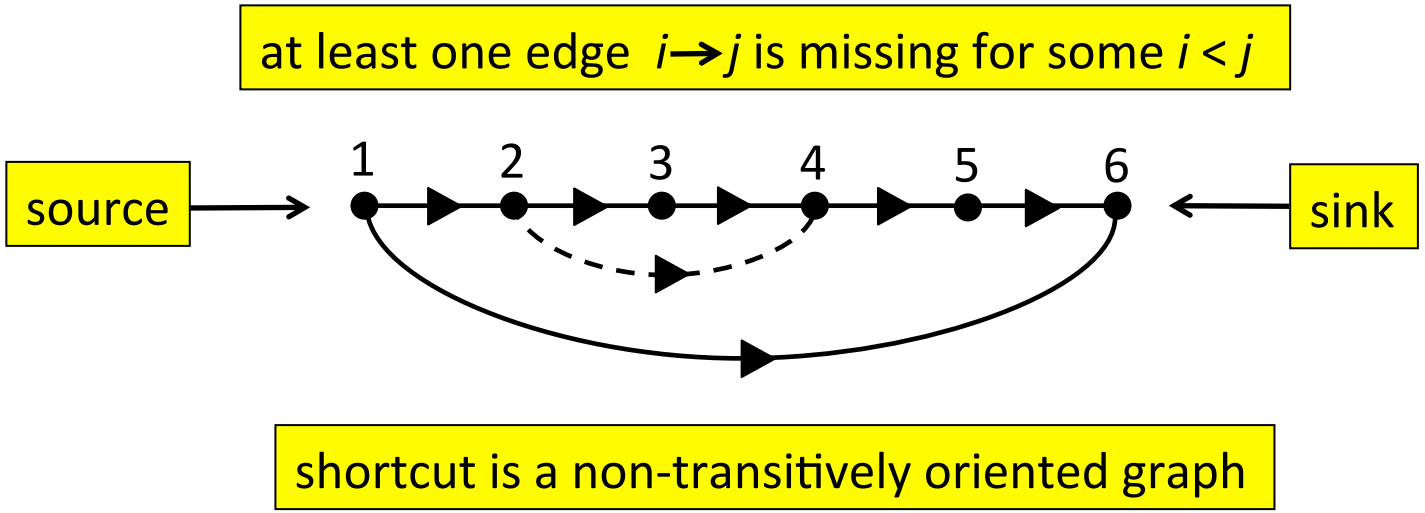


non-word-representable graph of maximum degree 4
(non-word-representable graphs constructed by adding an all-adjacent vertex contain a vertex of degree at least 5)

Triangle-free non-word-representable graphs exist too
(non-word-representable graphs constructed by adding an all-adjacent vertex contain triangles; so do non-word-representable graphs obtained from the last graph)

Characterization of word-representable graphs

Shortcut – a type of **acyclic** digraphs **on at least 4 vertices**

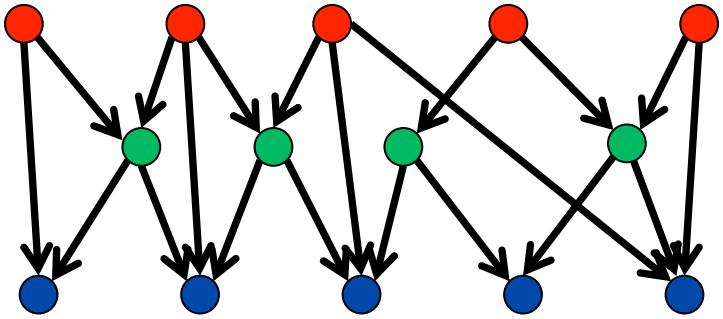


Semi-transitive orientation – acyclic shortcut-free orientation

Theorem (Halldórsson, SK, Pyatkin)

G is word-representable **iff** *G* accepts a semi-transitive orientation

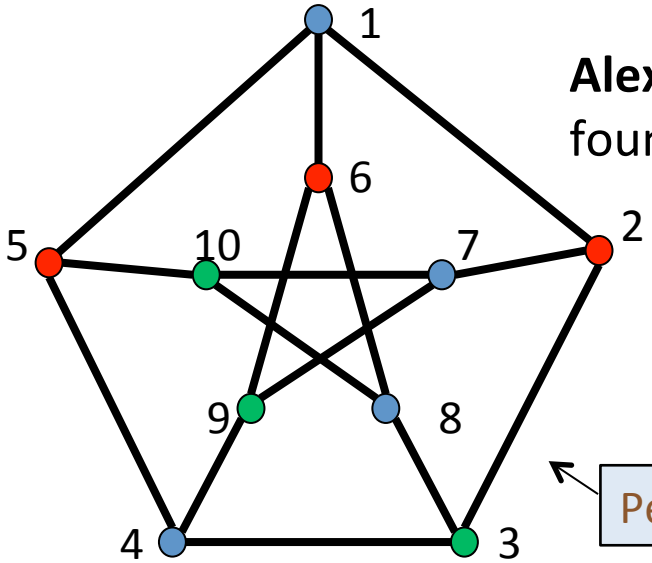
3-colorable graphs are word-representable



Red -> Green -> Blue: the orientation is shortcut-free

Most optimization problems on representable graphs are NP-hard

Independent Set, Dominating Set, Graph Coloring, Clique Partition...



Alexander Konovalov & **Steve Linton** (Univ. of St Andrews) found a 3-word-representation of the Petersen graph:

1 3 8 7 2 9 6 10 7 4 9 3 5 4 1 2 8 3 10 7 6 8 5 10 1 9 4 5 6 2

Petersen's graph

Various results

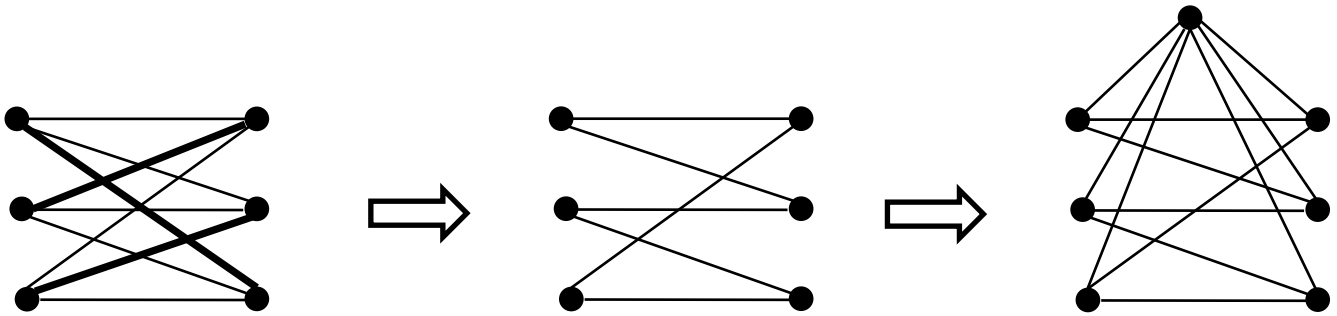
Theorem (Halldórsson, SK, Pyatkin)

Each word-representable graph on n vertices is n-word-representable

Thus the recognition problem of graph word-representability is in NP

Deciding whether a graph is k-representable for $2<k<n/2$ is NP-complete

There are graphs requiring $n/2$ copies of each letter, e.g. those based on a cocktail party graph

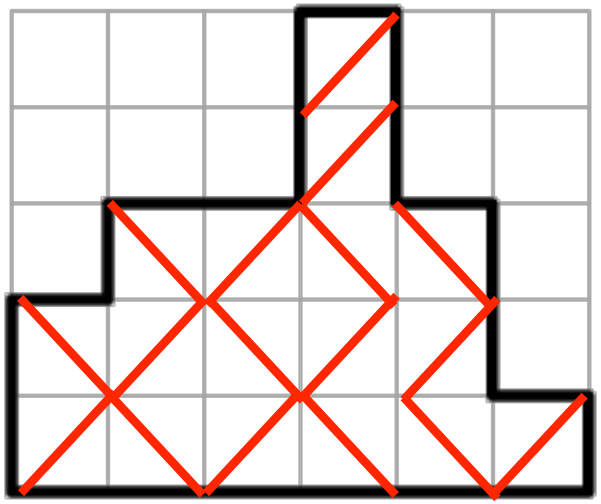


Asymptotics for the number of word-representable graphs

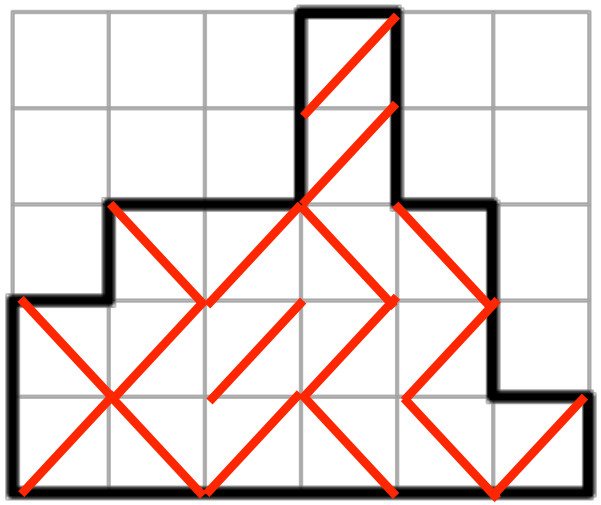
Theorem (Collins, SK, Lozin)

The number of word-representable graphs on *n* nodes is $2^{\frac{n^2}{3}+o(n^2)}$.

Triangulations of polyominoes



word-representable graph



non-word-representable graph

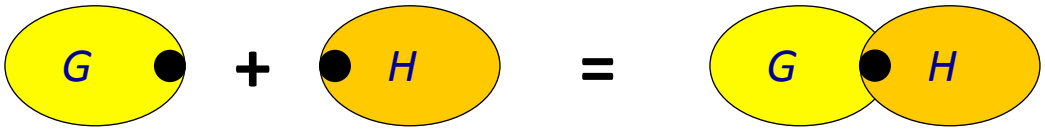
Theorem (Akrobotu, SK, Masarova)

A triangulation of a **convex** polyomino is word-representable **iff** it is 3-colorable.

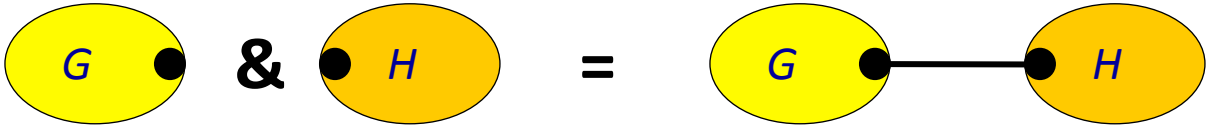
Graph operations preserving word-representability

- Replacing a node *v* by a module *H* (clique or any comparability graph);
  Neighbors of *v* become neighbors of all nodes in *H*

- Gluing two representable graphs in 1 node



- Joining two representable graphs by an edge



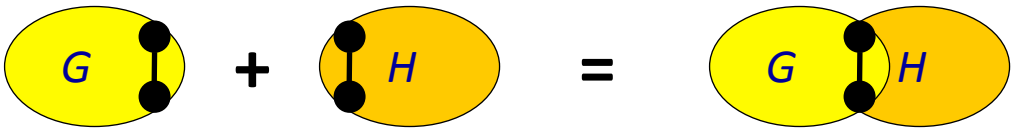
Graph operations **not** preserving word-representability

- Taking the line graph

Open question: Does it preserve non-word-representability? We only have examples of non-word-representable graphs going to non-word-representable graphs.

- Taking the complement

- Gluing two graphs at more than 1 node



Open questions

- Is it NP-hard to decide whether a given graph is word-representable?
- What is the maximum representation number of a graph (between $n/2$ and n)?
- Can we characterize the forbidden subgraphs of word-representable graphs?
- Characterize 3-word-representable graphs
- Is the line graph of a non-word-representable graph non-word-representable?
- How many (k-)word-representable (labeled) graphs are there?

Squares and cubes in words

factor in a word = a number of consecutive letters

square in a word = two consecutive equal factors

253253 is a square in the word 12532532

cube in a word = three consecutive equal factors

121212 is a cube in the word 11121212

trivial square in a word = two consecutive equal letters

The study of square-free words by **Axel Thue** (1863-1922) in the beginning of the 20th century originated the field of Combinatorics on Words (CoWs).

Square-free graph representations

Theorem: Cubes are avoidable in word-representants.

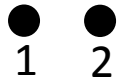
Proof: For a word X, $w = \ldots XXX \ldots$ and $w' = \ldots XX \ldots$ represent the same graph.

Question: Are squares avoidable in word-representants except for the case of independent set on two nodes? If not, give a classification of graphs whose word-representation requires presence of squares.

Trivial squares in word-representants

Lemma: If a word-representable graph contains at least one edge then the isolated nodes, if any, can be represented without involving trivial squares.

Proof: For isolated nodes *x*, *y*, *z*, … consider a word-representant … *zyx W xyz* …, where *W* represents the rest of the graph.

Theorem: Trivial squares are avoidable in representations of all but one graph (which is 

, the independent set on two nodes).

Proof: By the lemma, we need to represent properly an independent set on *n* nodes. For n>2 such representations are given by

2134 … (*n*-1) 12 … (*n*-1)*n*(*n*-1) … 21.

A general program of research

INPUT:

A language defined, e.g. through forbidden patterns

GOAL:

Study the class of graphs defined by the language (via alternations of letters in words)

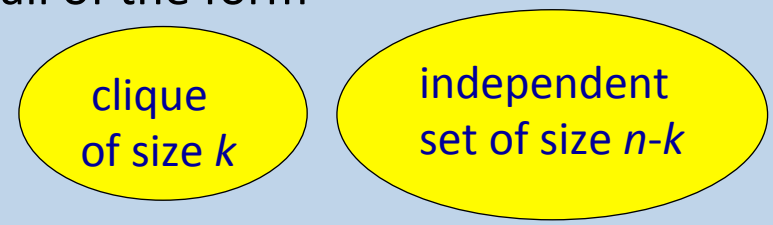
Trivial example:

INPUT:

Words avoiding the pattern 21 (classical or consecutive), that is, words of the form
11 … 122 … 233 … 3 … *nn* … *n*

OUTPUT:

There are n+1 unlabled graphs represented by this language all of the form



Do you have a clue what the output is for words avoiding longer patterns?!

Pattern-avoiding permutations and word-representability of graphs

INPUT:

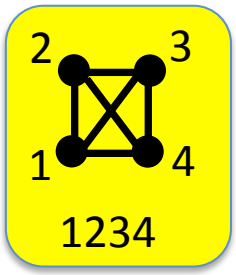
A pattern-avoiding class C

GOAL:

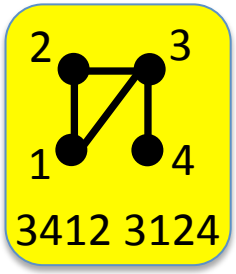
Study the class of graphs defined by concatenation of permutations from C.
Recall that such concatenations define comparability (transitively orientable) graphs.

Avoidance of a 2-letter pattern is trivial here just giving cliques as outcome.

For $Av(132)$, we at least can represent all **unlabeled** graphs on at most 4 nodes, e.g.



1234



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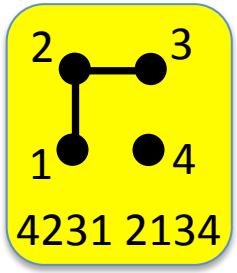

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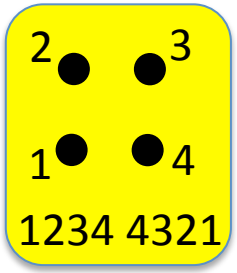

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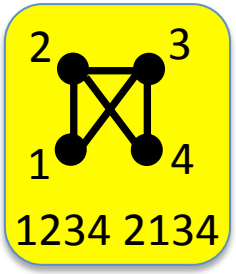
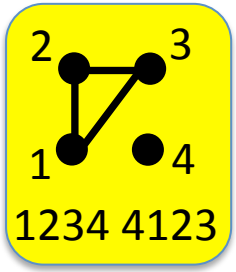


1234 2134



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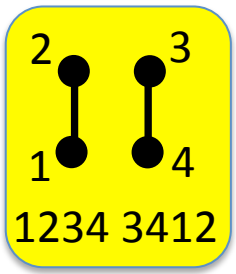

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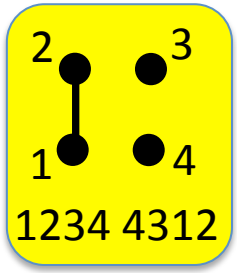

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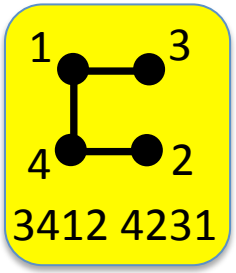

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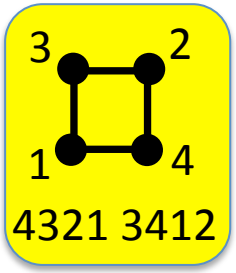

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THANK YOU FOR YOUR ATTENTION!