

Patterns and Statistics for Set Partitions

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Table of Contents

- Introduction
 - Pattern Avoidance in Set Partitions
 - Set Partitions and Restricted Growth Functions
 - Statistics on Restricted Growth Functions
 - Generating Functions
- General Results
 - Avoiding Patterns of Length 3
 - Avoiding Multiple Patterns of Length 3 and Patterns of Higher Length
- Selected Results
 - A Partial Characterization of $LB_n(123)$
 - Bijections
 - A q -analogue of the Catalan Numbers
- Closing Remarks

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For example, if $\sigma = 14/236/5$ then $\sigma' = 26/4$ is the subpartition obtained by intersecting the blocks of σ with $\{2, 4, 6\}$.

Pattern Containment and Avoidance

We define a standardization map $st(\sigma)$ that maps the smallest integer in the subpartition to 1, the second smallest to 2, etc.

We say σ *contains* π , which we call the *pattern*, if there is a subpartition σ' such that $st(\sigma') = \pi$. Otherwise we say σ *avoids* π .

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The cardinalities $|\Pi_n(\pi)|$ were determined by Sagan [2] for patterns for length = 3.

Standard Form and Restricted Growth Functions

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$$\min(B_1) < \min(B_2) < \dots < \min(B_k).$$

A *restricted growth function* (RGF) is a sequence of positive integers, $w = a_1 \dots a_n$, such that $a_1 = 1$ and for all $k \geq 2$,

$$a_k \leq 1 + \max(1, 2, \dots, a_{k-1}).$$

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For example, $137/24/56 \mapsto 1212331$.



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Define the lb statistic

$$lb(w) = \sum_i lb_i(w)$$

where $lb_i(w)$ is the number of integers to the left of w_i which are also bigger than w_i . (Note that multiple copies of the same integer are only counted once.)

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We can then use each statistic to create a generating function (polynomial). For lb ,

$$\text{LB}_n(\pi) = \text{LB}_n(\pi; q) = \sum_{\sigma \in \Pi_n(\pi)} q^{\text{lb}(w(\sigma))}.$$

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Note that the coefficient of q^k will be the number of partitions in the avoidance class of π that have an lb of k . Other statistics have polynomials defined in a similar manner.

Characterizations For 1/2/3

	Set Partition
Statistics	1/2/3
LB	$1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k$
LS	$\sum_{k=0}^{n-1} \binom{n-1}{k} q^k$
RB	$\sum_{k=0}^{n-1} \binom{n-1}{k} q^k$
RS	$1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k$

Characterizations For 1/23

	Set Partition
Statistics	1/23
LB	$1 + \sum_{k=0}^{n-2} (n - k - 1)q^k$
LS	$q^{T_{n-1}} + \sum_{i=0}^{n-2} (i + 1)q^{T_i}$
RB	$q^{T_{n-1}} + \sum_{j=1}^{n-1} \sum_{i=0}^{n-j-1} q^{j(n-j-1) + T_{n-j-2+i}}$
RS	$1 + \sum_{k=0}^{n-2} (n - k - 1)q^k$

Note that T_n is the n th triangular number, which is the sum $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Characterizations For 12/3

	Set Partition
Statistics	12/3
LB	$\sum_{k=0}^{\lfloor \frac{(n-1)^2}{4} \rfloor} D_k q^k$
LS	$q^{T_{n-1}} + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} q^{T_{n-j-1}+j(i-1)}$
RB	$q^{T_{n-1}} + \sum_{j=1}^{n-1} (n-j) q^{T_{n-j-1}}$
RS	$1 + \sum_{k=0}^{n-2} (n-1-k) q^k$

$$D_k = \#\{d : d \mid k \text{ and } d + \frac{k}{d} + 1 \leq n\}$$

Note that $D_k = \tau(k)$ for $k \leq n-2$.

Characterizations For $13/2$

	Set Partition
Statistics	$13/2$
LB	2^{n-1}
LS[1]	$\prod_{i=1}^{n-1} (1 + q^i)$
RB[1]	$\prod_{i=1}^{n-1} (1 + q^i)$
RS	2^{n-1}

Characterizations For 123

	Set Partition
Statistics	123
LB	Partial Characterization, See Below
LS	Partial Characterization
RB	Partial Characterization
RS	Partial Characterization

The polynomial $LB_n(123)$ has degree $\lfloor \frac{n(n-1)}{6} \rfloor$.

Avoiding Multiple Partition Patterns

Let P be a set of partitions. We let

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We have characterized all of the statistics for patterns of length 3 pair wise, as well as some of length four.

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First, note that $\sigma = B_1/\dots/B_k$ avoids 123 if and only if $|B_i| \leq 2$ for all i . So the corresponding RGF, w , has every entry repeated at most twice. Define the *initial run* of $w = a_1a_2\dots a_n$ to be the longest initial string of the form $12\dots i$.

Proving the Partial Characterization of $\text{LB}_n(123)$

We first show that a word of the form $w = 123 \dots i a_{i+1} \dots a_n$ with $a_{i+1} \dots a_n \leq i$ and $\{a_{i+1}, \dots, a_n\} = [n - i]$ maximizes the lb statistic.

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Proving the Partial Characterization of $LB_n(123)$

Now that we know that our word can be put into the form $w = 123 \dots i a_{i+1} \dots a_n$ with $a_{i+1} \dots a_n \leq i$ and $\{a_{i+1} \dots a_n\} = [n - i]$, we must pick the value of i to maximize lb .

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$$f(i) = \frac{(i-1) + (2i-n)}{2} (n-i) = \frac{-3i^2 + 4ni + i - n^2 - n}{2}.$$

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To maximize, we compute the derivative of f with respect to i to get $(-6i + 4n + 1)/2$ and set this equal to zero to obtain $i = (4n + 1)/6$. □

Proving the Partial Characterization of $LB_n(123)$

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for integer values of k . Finally, by substituting our values of i into the equation for the lb sum and using algebraic manipulations, we come up with the maximum value of lb being $\lfloor n(n-1)/6 \rfloor$. $\square$

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$$\phi : \Pi_n(\alpha) \mapsto \Pi_n(\beta)$$

with:

$$\phi(\pi) = \sigma \implies s(\pi) = t(\sigma).$$

Bijections

In looking at these avoidance classes, it can be interesting to find bijections between avoidance classes, $\Pi_n(\alpha)$ and $\Pi_n(\beta)$. More specifically, if $\pi \in \Pi_n(\alpha)$ and $\sigma \in \Pi_n(\beta)$, and s and t are two statistics on set partitions, we want a bijection

$$\phi : \Pi_n(\alpha) \mapsto \Pi_n(\beta)$$

with:

$$\phi(\pi) = \sigma \implies s(\pi) = t(\sigma).$$

Finding such a bijection between $\Pi_n(\alpha)$ and $\Pi_n(\beta)$ gives

$$\sum_{\pi \in \Pi_n(\alpha)} q^{s(\pi)} = \sum_{\sigma \in \Pi_n(\beta)} q^{t(\sigma)}. \quad (2)$$

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Theorem (D,D,G,G,P,R,S)

There exists a bijection $\phi : \Pi_n(1/2/3) \mapsto \Pi_n(1/2/3)$ that satisfies, $\forall \pi \in \Pi_n(1/2/3), \text{lb}(\pi) = \text{rs}(\phi(\pi))$.

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For Example

Let $\pi = 14/235 \in \Pi_n(1/2/3)$, with associated RGF

$$w = 12212.$$

$$\begin{array}{ccccccccc} w = & 1 & 2 & 2 & 1 & 2 & \mapsto & 1 & 1 & 2 & 1 & 1 & = w' \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{lb}(w) = & 0 & 0 & 0 & 1 & 0 & = 1 & = 0 & 0 & 1 & 0 & 0 & = \text{rs}(w') \end{array}$$

A q -analogue of the Catalan numbers

The partitions avoiding 13/24 are the noncrossing partitions. Therefore $|\Pi_n(13/24)| = C_n$. They are related to 2-colored Motzkin paths which are a Motzkin paths where each level step has been colored one of two colors.

Theorem (D,D,G,G,P,R,S)

Let $RS_n = RS_n(13/24)$. Then this polynomial satisfies the initial condition $RS_0 = 1$ and the recurrence

$$RS_n = 2 RS_{n-1} + \sum_{k=1}^{n-2} q^k RS_k RS_{n-k-1}.$$

This generating function is the same as the one for 2-colored Motzkin paths with n vertices by area.

A q -analogue of the Catalan numbers

The bijection used to prove this recurrence for $RS_n(13/24)$ is very similar to the direct sum of permutations and exactly the same as the bijection used by Chen, Dai, Dokos, Dwyer, and Sagan to prove a conjecture of Duncan and Steingrímsson on ascent sequences.

In future work, we hope to prove the following conjecture:

Conjecture

We have the equality

$$RS_n(13/24) = LB_n(13/24).$$

In Conclusion

Where to next?

- Patterns of Longer Length
- Multivariate Generating Functions
- 123
- 13/24
- Polynomials Generated by Other Statistics

Thank You! Any questions?

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