

Pattern Avoiding Involutions and the q -Analogues for Binomial Coefficients

Samantha Dahlberg Bruce Sagan

Michigan State University

July 8, 2014

Starter Notation

Let $\mathfrak{S}_n$ be the *symmetric group* and for $\pi \in \mathfrak{S}_k$

$$\mathfrak{S}_n(\pi) = \{\omega \in \mathfrak{S}_n : \omega \text{ avoids } \pi\}.$$

Starter Notation

Let $\mathfrak{S}_n$ be the *symmetric group* and for $\pi \in \mathfrak{S}_k$

$$\mathfrak{S}_n(\pi) = \{\omega \in \mathfrak{S}_n : \omega \text{ avoids } \pi\}.$$

An *involution* is a permutation $\iota \in \mathfrak{S}_n$ which has cycles of length one or two.

Starter Notation

Let $\mathfrak{S}_n$ be the *symmetric group* and for $\pi \in \mathfrak{S}_k$

$$\mathfrak{S}_n(\pi) = \{\omega \in \mathfrak{S}_n : \omega \text{ avoids } \pi\}.$$

An *involution* is a permutation $\iota \in \mathfrak{S}_n$ which has cycles of length one or two. For *example*

$$341256879 = (1, 3)(2, 4)(5)(6)(7, 8)(9)$$

is an involution.

Starter Notation

Let $\mathfrak{S}_n$ be the *symmetric group* and for $\pi \in \mathfrak{S}_k$

$$\mathfrak{S}_n(\pi) = \{\omega \in \mathfrak{S}_n : \omega \text{ avoids } \pi\}.$$

An *involution* is a permutation $\iota \in \mathfrak{S}_n$ which has cycles of length one or two. For *example*

$$341256879 = (1, 3)(2, 4)(5)(6)(7, 8)(9)$$

is an involution.

Let $\mathcal{I}_n = \{\iota \in \mathfrak{S}_n : \iota \text{ is an involution}\}$ and

$$\mathcal{I}_n(\pi) = \{\iota \in \mathcal{I}_n : \iota \text{ avoids } \pi\}.$$

Wilf Equivalence

Permutations $\pi, \tau \in \mathfrak{S}_k$ are *Wilf equivalent* if $|\mathfrak{S}_n(\pi)| = |\mathfrak{S}_n(\tau)|$.

$$|\mathfrak{S}_n(\pi)| = \frac{1}{n+1} \binom{2n}{n}, \forall \pi \in \mathfrak{S}_3.$$

Wilf Equivalence

Permutations $\pi, \tau \in \mathfrak{S}_k$ are *Wilf equivalent* if $|\mathfrak{S}_n(\pi)| = |\mathfrak{S}_n(\tau)|$.

$$|\mathfrak{S}_n(\pi)| = \frac{1}{n+1} \binom{2n}{n}, \forall \pi \in \mathfrak{S}_3.$$

Permutations $\pi, \tau \in \mathcal{I}_k$ are *$\mathcal{I}$ -Wilf equivalent* if $|\mathcal{I}_n(\pi)| = |\mathcal{I}_n(\tau)|$.

Wilf Equivalence

Permutations $\pi, \tau \in \mathfrak{S}_k$ are *Wilf equivalent* if $|\mathfrak{S}_n(\pi)| = |\mathfrak{S}_n(\tau)|$.

$$|\mathfrak{S}_n(\pi)| = \frac{1}{n+1} \binom{2n}{n}, \forall \pi \in \mathfrak{S}_3.$$

Permutations $\pi, \tau \in \mathcal{I}_k$ are *$\mathcal{I}$ -Wilf equivalent* if $|\mathcal{I}_n(\pi)| = |\mathcal{I}_n(\tau)|$.

Theorem (Simion-Schmidt)

There are two $\mathcal{I}$ -Wilf equivalent classes in $\mathfrak{S}_3$.

$$|\mathcal{I}_n(\pi)| = \binom{n}{\lfloor n/2 \rfloor} \quad \text{for } \pi = 123, 132, 321, 213$$

$$|\mathcal{I}_n(\pi)| = 2^{n-1} \quad \text{for } \pi = 231, 312$$

More Wilf Equivalence

Let us throw in some statistics!

More Wilf Equivalence

Let us throw in some statistics! The *major index* of a sequence of integers $\pi = a_1 a_2 \dots a_n$ is

$$\text{maj}(\pi) = \sum_{a_i > a_{i+1}} i.$$

More Wilf Equivalence

Let us throw in some statistics! The *major index* of a sequence of integers $\pi = a_1 a_2 \dots a_n$ is

$$\text{maj}(\pi) = \sum_{a_i > a_{i+1}} i.$$

Let

$$M\mathcal{I}_n(\pi; q) = M\mathcal{I}_n(\pi) = \sum_{\iota \in \mathcal{I}_n(\pi)} q^{\text{maj}(\iota)}.$$

More Wilf Equivalence

Let us throw in some statistics! The *major index* of a sequence of integers $\pi = a_1 a_2 \dots a_n$ is

$$\text{maj}(\pi) = \sum_{a_i > a_{i+1}} i.$$

Let

$$M\mathcal{I}_n(\pi; q) = M\mathcal{I}_n(\pi) = \sum_{\iota \in \mathcal{I}_n(\pi)} q^{\text{maj}(\iota)}.$$

We say that π and τ are *$M\mathcal{I}$ -Wilf equivalent* if

$$M\mathcal{I}_n(\pi) = M\mathcal{I}_n(\tau).$$

More Wilf Equivalence

Let us throw in some statistics! The *major index* of a sequence of integers $\pi = a_1 a_2 \dots a_n$ is

$$\text{maj}(\pi) = \sum_{a_i > a_{i+1}} i.$$

Let

$$MI_n(\pi; q) = MI_n(\pi) = \sum_{\iota \in \mathcal{I}_n(\pi)} q^{\text{maj}(\iota)}.$$

We say that π and τ are *MI-Wilf equivalent* if

$$MI_n(\pi) = MI_n(\tau).$$

Theorem (DS)

There are five MI-Wilf equivalence in $\mathfrak{S}_3$:

$\{231, 312\}$, $\{213\}$, $\{132\}$, $\{123\}$ and $\{321\}$.

q -Analogues

A q -analogue of a number N is a polynomial $f(q)$ such that if we let $q = 1$ we have $f(1) = N$.

q -Analogues

A q -analogue of a number N is a polynomial $f(q)$ such that if we let $q = 1$ we have $f(1) = N$. Some q -analogues are considered *standard q -analogues*.

q -Analogues

A q -analogue of a number N is a polynomial $f(q)$ such that if we let $q = 1$ we have $f(1) = N$. Some q -analogues are considered *standard q -analogues*.

- The standard q -analogue for a positive integer n is

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$$

$$[n]_1 = 1 + 1 + 1^2 + \cdots + 1^{n-1} = n.$$

q -Analogues

A q -analogue of a number N is a polynomial $f(q)$ such that if we let $q = 1$ we have $f(1) = N$. Some q -analogues are considered *standard q -analogues*.

- The standard q -analogue for a positive integer n is

$$\begin{aligned}[n]_q &= 1 + q + q^2 + \cdots + q^{n-1} \\ [n]_1 &= 1 + 1 + 1^2 + \cdots + 1^{n-1} = n.\end{aligned}$$

- The standard q -analogue for $n!$ is

$$[n]_q! = [1]_q [2]_q \cdots [n]_q.$$

q -Analogues

A q -analogue of a number N is a polynomial $f(q)$ such that if we let $q = 1$ we have $f(1) = N$. Some q -analogues are considered *standard q -analogues*.

- The standard q -analogue for a positive integer n is

$$\begin{aligned}[n]_q &= 1 + q + q^2 + \cdots + q^{n-1} \\ [n]_1 &= 1 + 1 + 1^2 + \cdots + 1^{n-1} = n.\end{aligned}$$

- The standard q -analogue for $n!$ is

$$[n]_q! = [1]_q [2]_q \cdots [n]_q.$$

- The standard q -analogue for $\binom{n}{k}$ is

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}.$$

q -Analogues

For $\pi \in \{123, 321, 213, 132\}$ we have $|\mathcal{I}_n(\pi)| = \binom{n}{\lfloor n/2 \rfloor}$.

q -Analogues

For $\pi \in \{123, 321, 213, 132\}$ we have $|\mathcal{I}_n(\pi)| = \binom{n}{\lfloor n/2 \rfloor}$. Which means for $M\mathcal{I}_n(\pi)$ we have

$$M\mathcal{I}_n(\pi; 1) = \sum_{\iota \in \mathcal{I}_n(\pi)} 1^{\text{maj}(\iota)} = \binom{n}{\lfloor n/2 \rfloor}.$$

q -Analogues

For $\pi \in \{123, 321, 213, 132\}$ we have $|\mathcal{I}_n(\pi)| = \binom{n}{\lfloor n/2 \rfloor}$. Which means for $M\mathcal{I}_n(\pi)$ we have

$$M\mathcal{I}_n(\pi; 1) = \sum_{\iota \in \mathcal{I}_n(\pi)} 1^{\text{maj}(\iota)} = \binom{n}{\lfloor n/2 \rfloor}.$$

So $M\mathcal{I}_n(\pi)$ is a q -analogue for $\binom{n}{\lfloor n/2 \rfloor}$ for $\pi \in \{123, 321, 213, 132\}$.

q -Analogues

For $\pi \in \{123, 321, 213, 132\}$ we have $|\mathcal{I}_n(\pi)| = \binom{n}{\lfloor n/2 \rfloor}$. Which means for $M\mathcal{I}_n(\pi)$ we have

$$M\mathcal{I}_n(\pi; 1) = \sum_{\iota \in \mathcal{I}_n(\pi)} 1^{\text{maj}(\iota)} = \binom{n}{\lfloor n/2 \rfloor}.$$

So $M\mathcal{I}_n(\pi)$ is a q -analogue for $\binom{n}{\lfloor n/2 \rfloor}$ for $\pi \in \{123, 321, 213, 132\}$.

Question: Are any of these the standard q -analogue?

q -Analogues

For $\pi \in \{123, 321, 213, 132\}$ we have $|\mathcal{I}_n(\pi)| = \binom{n}{\lfloor n/2 \rfloor}$. Which means for $M\mathcal{I}_n(\pi)$ we have

$$M\mathcal{I}_n(\pi; 1) = \sum_{\iota \in \mathcal{I}_n(\pi)} 1^{\text{maj}(\iota)} = \binom{n}{\lfloor n/2 \rfloor}.$$

So $M\mathcal{I}_n(\pi)$ is a q -analogue for $\binom{n}{\lfloor n/2 \rfloor}$ for $\pi \in \{123, 321, 213, 132\}$.

Question: Are any of these the standard q -analogue?

Answer: Yes!

q -Analogues

For $\pi \in \{123, 321, 213, 132\}$ we have $|\mathcal{I}_n(\pi)| = \binom{n}{\lfloor n/2 \rfloor}$. Which means for $M\mathcal{I}_n(\pi)$ we have

$$M\mathcal{I}_n(\pi; 1) = \sum_{\iota \in \mathcal{I}_n(\pi)} 1^{\text{maj}(\iota)} = \binom{n}{\lfloor n/2 \rfloor}.$$

So $M\mathcal{I}_n(\pi)$ is a q -analogue for $\binom{n}{\lfloor n/2 \rfloor}$ for $\pi \in \{123, 321, 213, 132\}$.

Question: Are any of these the standard q -analogue?

Answer: Yes!

Theorem (DS)

We have the following equality of q -analogues

$$M\mathcal{I}_n(321; q) = \left[\begin{matrix} n \\ \lceil n/2 \rceil \end{matrix} \right]_q.$$

Proof

We want to show $MI_n(321) = \left[\begin{smallmatrix} n \\ \lceil n/2 \rceil \end{smallmatrix} \right]_q$.

Proof

We want to show $MI_n(321) = \left[\begin{smallmatrix} n \\ \lceil n/2 \rceil \end{smallmatrix} \right]_q$.

Proposition

For $W_{n,k}$ the set of length n words with k ones and $n - k$ zeros we have

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = \sum_{w \in W_{n,k}} q^{\text{maj}(w)}.$$

Proof

We want to show $MI_n(321) = \left[\begin{smallmatrix} n \\ \lceil n/2 \rceil \end{smallmatrix} \right]_q$.

Proposition

For $W_{n,k}$ the set of length n words with k ones and $n - k$ zeros we have

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = \sum_{w \in W_{n,k}} q^{\text{maj}(w)}.$$

Idea:

- Show for $k = \lceil \frac{n}{2} \rceil$

$$MI_n(321) = \sum_{\iota \in \mathcal{I}_n(321)} q^{\text{maj}(\iota)} = \sum_{w \in W_{n,k}} q^{\text{maj}(w)}.$$

Proof

We want to show $MI_n(321) = \left[\begin{smallmatrix} n \\ \lceil n/2 \rceil \end{smallmatrix} \right]_q$.

Proposition

For $W_{n,k}$ the set of length n words with k ones and $n - k$ zeros we have

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = \sum_{w \in W_{n,k}} q^{\text{maj}(w)}.$$

Idea:

- Show for $k = \lceil \frac{n}{2} \rceil$

$$MI_n(321) = \sum_{\iota \in \mathcal{I}_n(321)} q^{\text{maj}(\iota)} = \sum_{w \in W_{n,k}} q^{\text{maj}(w)}.$$

- Want a bijection $\phi : \mathcal{I}_n(321) \rightarrow W_{n,k}$ for $k = \lceil \frac{n}{2} \rceil$.

Proof

We want to show $M\mathcal{I}_n(321) = \left[\begin{smallmatrix} n \\ \lceil n/2 \rceil \end{smallmatrix} \right]_q$.

Proposition

For $W_{n,k}$ the set of length n words with k ones and $n - k$ zeros we have

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = \sum_{w \in W_{n,k}} q^{\text{maj}(w)}.$$

Idea:

- Show for $k = \lceil \frac{n}{2} \rceil$

$$M\mathcal{I}_n(321) = \sum_{\iota \in \mathcal{I}_n(321)} q^{\text{maj}(\iota)} = \sum_{w \in W_{n,k}} q^{\text{maj}(w)}.$$

- Want a bijection $\phi : \mathcal{I}_n(321) \rightarrow W_{n,k}$ for $k = \lceil \frac{n}{2} \rceil$.
- Want ϕ to preserve the major index, $\text{maj}(\phi(\iota)) = \text{maj}(\iota)$.

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

$$341256879 \xrightarrow{\phi}$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .

$$341256879 \xrightarrow{\phi}$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .

$$341256879 \xrightarrow{\phi} 1 \ 0$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .

$$341256879 \xrightarrow{\phi} 1100$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .

$$341256\textcolor{red}{8}79 \xrightarrow{\phi} 1100 \quad 10$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .
- For fixed points place required number of 0's and 1's in weakly increasing order so we have $\lceil \frac{n}{2} \rceil$ ones and $\lfloor \frac{n}{2} \rfloor$ zeros.

$$341256\textcolor{red}{8}79 \xrightarrow{\phi} 1100 \quad 10$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .
- For fixed points place required number of 0's and 1's in weakly increasing order so we have $\lceil \frac{n}{2} \rceil$ ones and $\lfloor \frac{n}{2} \rfloor$ zeros.

For our **example** involution $n = 9$ and $\lceil \frac{n}{2} \rceil = 5$.

$$341256\mathbf{87}9 \xrightarrow{\phi} 1100 \quad 10$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .
- For fixed points place required number of 0's and 1's in weakly increasing order so we have $\lceil \frac{n}{2} \rceil$ ones and $\lfloor \frac{n}{2} \rfloor$ zeros.

For our **example** involution $n = 9$ and $\lceil \frac{n}{2} \rceil = 5$. Our word needs 5 ones and 4 zeros. We place 2 ones and 1 zero in weakly increasing order.

$$341256\textcolor{red}{87}9 \xrightarrow{\phi} 1100 \quad 10$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .
- For fixed points place required number of 0's and 1's in weakly increasing order so we have $\lceil \frac{n}{2} \rceil$ ones and $\lfloor \frac{n}{2} \rfloor$ zeros.

For our **example** involution $n = 9$ and $\lceil \frac{n}{2} \rceil = 5$. Our word needs 5 ones and 4 zeros. We place 2 ones and 1 zero in weakly increasing order.

$$3412\textcolor{red}{56879} \xrightarrow{\phi} 110001101$$

The map of ϕ

Given $\iota \in \mathcal{I}_n(321)$ we define ϕ as follows.

- For 2-cycles (i, j) with $i < j$ place 1 in position i and 0 in position j .
- For fixed points place required number of 0's and 1's in weakly increasing order so we have $\lceil \frac{n}{2} \rceil$ ones and $\lfloor \frac{n}{2} \rfloor$ zeros.
- The map ϕ is well defined since the word has $\lceil \frac{n}{2} \rceil$ ones and $\lfloor \frac{n}{2} \rfloor$ zeros.

For our **example** involution $n = 9$ and $\lceil \frac{n}{2} \rceil = 5$. Our word needs 5 ones and 4 zeros. We place 2 ones and 1 zero in weakly increasing order.

$$341256879 \xrightarrow{\phi} 110001101$$

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

)))(((

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

We start with a string of left and right parentheses. The *core* is a subset of these parentheses determined the following way.

$)()((()())((()())(($

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

We start with a string of left and right parentheses. The *core* is a subset of these parentheses determined the following way.

- Match adjacent left and right parentheses.

))(((())((((())((

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

We start with a string of left and right parentheses. The *core* is a subset of these parentheses determined the following way.

- Match adjacent left and right parentheses.

)) (((()) (((()) ((

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

We start with a string of left and right parentheses. The *core* is a subset of these parentheses determined the following way.

- Match adjacent left and right parentheses.
- As if previously matched parentheses are ignored, continue matching remaining parentheses.

))((**(**)**(**)(((**(**)((

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

We start with a string of left and right parentheses. The *core* is a subset of these parentheses determined the following way.

- Match adjacent left and right parentheses.
- As if previously matched parentheses are ignored, continue matching remaining parentheses.

))((()())((()())((

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

We start with a string of left and right parentheses. The *core* is a subset of these parentheses determined the following way.

- Match adjacent left and right parentheses.
- As if previously matched parentheses are ignored, continue matching remaining parentheses.
- You are done when there are no more possible matchings.

))((()())((()())((

The Core

This concept is due to Greene and Kleitman, and has been used in finding symmetric chain decomposition for posets.

We start with a string of left and right parentheses. The *core* is a subset of these parentheses determined the following way.

- Match adjacent left and right parentheses.
- As if previously matched parentheses are ignored, continue matching remaining parentheses.
- You are done when there are no more possible matchings.
- The set of matched parentheses is the core.

))((())((())((

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

$$110001101 \xrightarrow{\phi^{-1}}$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.

$$\begin{array}{c} 110001101 \xrightarrow{\phi^{-1}} \\ ((()))((()) \end{array}$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.

$$110001101 \xrightarrow{\phi^{-1}} ((()))((() ($$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.

$$110001101 \xrightarrow{\phi^{-1}} \\ ((()))((() ($$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.

$$\begin{array}{c} \underline{110001101} \xrightarrow{\phi^{-1}} \\ ((()))((() \end{array}$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.
- Inside the core, the indices of the k th 1 and k th 0 become the transpositions.

$$\underline{110001}\underline{101} \xrightarrow{\phi^{-1}}$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.
- Inside the core, the indices of the k th 1 and k th 0 become the transpositions.

$$\underline{1100}01\underline{10}1 \xrightarrow{\phi^{-1}} 3412 \quad 87$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.
- Inside the core, the indices of the k th 1 and k th 0 become the transpositions.
- The remaining indices become the fixed points.

$$\underline{1100}01\underline{10}1 \xrightarrow{\phi^{-1}} 3412 \quad 87$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.
- Inside the core, the indices of the k th 1 and k th 0 become the transpositions.
- The remaining indices become the fixed points.

$$\underline{1100}01\underline{10}1 \xrightarrow{\phi^{-1}} 341256879$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.
- Inside the core, the indices of the k th 1 and k th 0 become the transpositions.
- The remaining indices become the fixed points.
- The positions of the 1's and 0's each correspond to an increasing sequence in ι . Hence the longest decreasing sequence in ι is at most 2.

$$\underline{1100}01\underline{10}1 \xrightarrow{\phi^{-1}} 341256879$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.
- Inside the core, the indices of the k th 1 and k th 0 become the transpositions.
- The remaining indices become the fixed points.
- The positions of the 1's and 0's each correspond to an increasing sequence in ι . Hence the longest decreasing sequence in ι is at most 2.

$$\underline{1100}01\underline{10}1 \xrightarrow{\phi^{-1}} 341256879$$

$$110001101 \xrightarrow{\phi^{-1}} 341256879$$

The map of ϕ^{-1}

Given $w \in W_{n,k}$ with $k = \lceil \frac{n}{2} \rceil$ we define ϕ^{-1} as follows.

- Consider all the 1's as left parentheses and all the 0's as right parentheses. Determine the core.
- Inside the core, the indices of the k th 1 and k th 0 become the transpositions.
- The remaining indices become the fixed points.
- The positions of the 1's and 0's each correspond to an increasing sequence in ι . Hence the longest decreasing sequence in ι is at most 2.
- The resulting transposition ι avoids 321, so our map is well defined.

$$\underline{1100}01\underline{10}1 \xrightarrow{\phi^{-1}} 341256879$$

$$110001101 \xrightarrow{\phi^{-1}} 341256879$$

Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani

They have independently shown $M\mathcal{I}_n(321) = \left[\begin{smallmatrix} n \\ \lfloor n/2 \rfloor \end{smallmatrix} \right]_q$ by using that

$$\left[\begin{smallmatrix} n \\ \lfloor n/2 \rfloor \end{smallmatrix} \right]_q = \sum_{\lambda \subseteq B_n} q^{|\lambda|}$$

where B_n is a $\lfloor n/2 \rfloor \times \lceil n/2 \rceil$ box.

Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani

They have independently shown $M\mathcal{I}_n(321) = \left[\begin{smallmatrix} n \\ \lfloor n/2 \rfloor \end{smallmatrix} \right]_q$ by using that

$$\left[\begin{smallmatrix} n \\ \lfloor n/2 \rfloor \end{smallmatrix} \right]_q = \sum_{\lambda \subseteq B_n} q^{|\lambda|}$$

where B_n is a $\lfloor n/2 \rfloor \times \lceil n/2 \rceil$ box. They use a bijection

$$\begin{aligned} \iota &\longrightarrow \lambda \\ \text{Des}(\iota) &\longrightarrow \text{HD}(\lambda) \end{aligned}$$

where $\text{Des}(\iota) = \{i_1, \dots, i_s\}$ is the descent set and $\text{HD}(\lambda)$ is called the hook decomposition of λ : i_s is the largest hook length of λ , i_{s-1} is the largest hook length of λ after removing the largest hook, and so on.

Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani

An **example**:

For $\iota = 341256879$

Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani

An **example**:

For $\iota = 341256879$ we have $\text{Des}(\iota) = \{2, 7\}$

Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani

An **example**:

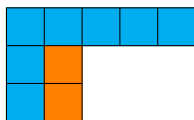
For $\iota = 341256879$ we have $\text{Des}(\iota) = \{2, 7\}$ so $\text{maj}(\iota) = 9$.

Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani

An **example**:

For $\iota = 341256879$ we have $\text{Des}(\iota) = \{2, 7\}$ so $\text{maj}(\iota) = 9$.

This will be in bijection with $\lambda = \{5, 2, 2\}$

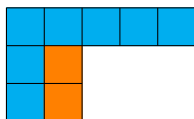


Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani

An **example**:

For $\iota = 341256879$ we have $\text{Des}(\iota) = \{2, 7\}$ so $\text{maj}(\iota) = 9$.

This will be in bijection with $\lambda = \{5, 2, 2\}$ with $\text{HD}(\lambda) = \{2, 7\}$ and $|\lambda| = 9$.



q -Analogue for Binomial Coefficients

What about $\begin{bmatrix} n \\ k \end{bmatrix}_q$?

q -Analogue for Binomial Coefficients

What about $\begin{bmatrix} n \\ k \end{bmatrix}_q$?

Theorem (DS)

Let $t(\iota)$ be the number of transpositions in ι . We have the following equality of q -analogues for $k \leq \lfloor \frac{n}{2} \rfloor$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\iota \in \mathcal{I}_n(321) \\ t(\iota) \leq k}} q^{\text{maj}(\iota)}.$$

q -Analogue for Binomial Coefficients

What about $\begin{bmatrix} n \\ k \end{bmatrix}_q$?

Theorem (DS)

Let $t(\iota)$ be the number of transpositions in ι . We have the following equality of q -analogues for $k \leq \lfloor \frac{n}{2} \rfloor$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\iota \in \mathcal{I}_n(321) \\ t(\iota) \leq k}} q^{\text{maj}(\iota)}.$$

In the proof we modify the required number of ones and zeros in the word.

Example:

$$MI_4(123) = q^2 + q^3 + 2q^4 + q^5 + q^6$$

Example:

$$MI_4(123) = q^2 + q^3 + 2q^4 + q^5 + q^6$$

$$MI_4(321) = 1 + q + 2q^2 + q^3 + q^4$$

Example:

$$MI_4(123) = q^2 + q^3 + 2q^4 + q^5 + q^6$$

$$MI_4(321) = 1 + q + 2q^2 + q^3 + q^4$$

Theorem (DS)

We have the following equalities for $MI_n(123)$.

$$MI_n(123; q) = q^{\binom{n}{2}} \mathcal{I}_n(321; q^{-1}) = q^N \left[\begin{matrix} n \\ \lceil n/2 \rceil \end{matrix} \right]_q$$

where $N = \binom{n}{2} - \lfloor n/2 \rfloor \lceil n/2 \rceil$.

$MI_n(213)$

Examples:

$$MI_3(213) = 1 + q^2 + q^3$$

$$MI_4(213) = 1 + q^2 + q^3 + q^4 + q^5 + q^6$$

Examples:

$$M\mathcal{I}_3(213) = 1 + q^2 + q^3$$

$$M\mathcal{I}_4(213) = 1 + q^2 + q^3 + q^4 + q^5 + q^6$$

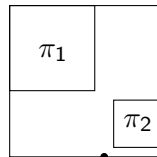
Theorem (DS)

If $\iota \in \mathcal{I}_n(213)$ then

- *$\text{maj}(\iota) = 0$ or $\text{maj}(\iota) \geq \lceil n/2 \rceil$*
- *this bound is sharp*
- *for every k with $\lceil n/2 \rceil \leq k \leq \binom{n}{2}$ there exists some $\iota \in \mathcal{I}_n(213)$ with $\text{maj}(\iota) = k$.*

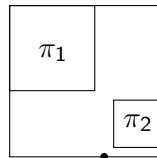
Proof

For a permutation which avoids 213 its diagram is of the form at right with π_1 and π_2 both avoiding 213.

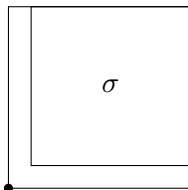
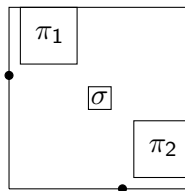


Proof

For a permutation which avoids 213 its diagram is of the form at right with π_1 and π_2 both avoiding 213.

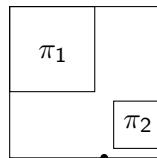


Involutions $\iota \in \mathcal{I}_n(213)$ have the two possible forms determined by whether $\iota(1) = 1$ or $\iota(1) \neq 1$, due to diagonal symmetry.

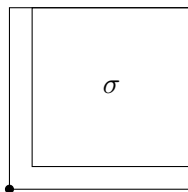
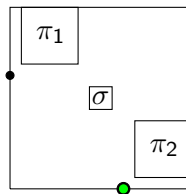


Proof

For a permutation which avoids 213 its diagram is of the form at right with π_1 and π_2 both avoiding 213.



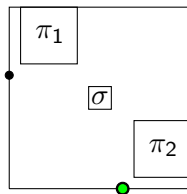
Involutions $\iota \in \mathcal{I}_n(213)$ have the two possible forms determined by whether $\iota(1) = 1$ or $\iota(1) \neq 1$, due to diagonal symmetry.



In the case on the left π_1 and π_2 are of the same size so $\iota(1) \geq \lceil n/2 \rceil + 1$.

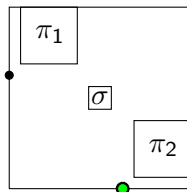
Proof

Here are our second case again.



Proof

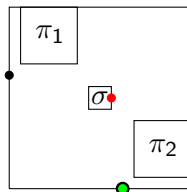
Here are our second case again.



- We have green dot $\geq \lceil n/2 \rceil + 1$.

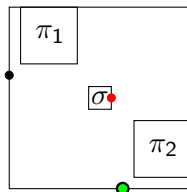
Proof

Here are our second case again.



- We have **green** dot $\geq \lceil n/2 \rceil + 1$.
- We are forced to have one decent at the **red** dot right before the green dot.

Here are our second case again.



- We have **green** dot $\geq \lceil n/2 \rceil + 1$.
- We are forced to have one decent at the **red** dot right before the green dot.
- So $\text{maj}(\iota) \geq \text{index of red dot} \geq \lceil n/2 \rceil$.

Theorem (DS)

If $\iota \in \mathcal{I}_n(132)$ then

- *$\text{maj}(\iota) = \binom{n}{2}$ or $\text{maj}(\iota) \leq \binom{n}{2} - \lceil n/2 \rceil$*
- *this bound is sharp*
- *for every non-negative $k \leq \binom{n}{2} - \lceil n/2 \rceil$ there exists some $\iota \in \mathcal{I}_n(132)$ with $\text{maj}(\iota) = k$.*

Theorem (DS)

If $\iota \in \mathcal{I}_n(132)$ then

- *$\text{maj}(\iota) = \binom{n}{2}$ or $\text{maj}(\iota) \leq \binom{n}{2} - \lceil n/2 \rceil$*
- *this bound is sharp*
- *for every non-negative $k \leq \binom{n}{2} - \lceil n/2 \rceil$ there exists some $\iota \in \mathcal{I}_n(132)$ with $\text{maj}(\iota) = k$.*

Theorem (DS)

We have the following equivalence.

$$q^{\binom{n}{2}} \mathcal{I}_n(132; q^{-1}) = \mathcal{I}_n(213; q).$$

References

- ① Marilena Barnabei, Flavio Bonetti, Sergi Elizalde and Matteo Silimbani, Descent sets on 321-avoiding involutions and hook decompositions of partitions, arXiv:1401.3011.
- ② Janet Beissinger, Similar constructions for Young tableaux and involutions, and their application to shiftable tableaux, *Discrete Mathematics*, **67** (1987) 149-163.
- ③ Theodore Dokos, Tim Dwyer, Bryan Johnson, Bruce Sagan and Kimberly Selsor, Permutation patterns and statistics, *Discrete Mathematics*, **312** (2013) 2760-2775.
- ④ Curtis Greene and Daniel Kleitman, Strong versions of Sperner's Theorem, *J. Combinatorial Theory Ser. A*, **20** (1976), no. 1, 80-88.
- ⑤ Rodica Simion and Frank Schmidt, Restricted permutations, *Europ. J. Combinatorics*, **6** (1985) 383-406.

The End

Thank You For **Listening!**