Functional equations for enumerating 1324-avoiding permutations

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Permutation Patterns 2014

Upcoming...

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DIMACS Conference on Challenges of Identifying Integer Sequences
Celebrating 50 years of the OEIS and Neil Sloane's 75th birthday!
October 9–10, 2014

Poster abstracts are welcomed!

Some funding may be available for "early career" researchers to attend.

For more details, see: http://dimacs.rutgers.edu/Workshops/OEIS/

Pattern occurrences

Definition

Given a (permutation) pattern $\tau = \tau_1 \dots \tau_k$, we say that permutation $\pi = \pi_1 \dots \pi_n$ *contains* the pattern τ if there exists $1 \leq i_1 < \dots < i_k \leq n$ such that $\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}$ is order-isomorphic to τ.

Example

If pattern $\tau = 123$,

- $\pi = 54321$ has zero occurrences of τ,
- $\pi = 42135$ has two occurrences of τ.

Some notation

Definition

Given a pattern $\tau \in \mathcal{S}_k$, define

$$\mathcal{S}_n(\tau) := \{\pi \in \mathcal{S}_n : \pi \text{ avoids } \tau\},$$
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Definition

Given a pattern τ and $r \geq 0$, define

$$\mathcal{S}_n(\tau, r) := \{\pi \in \mathcal{S}_n : \pi \text{ with exactly } r \text{ occurrences of } \tau\},$$
$$s_n(\tau, r) := |\mathcal{S}_n(\tau, r)|.$$

Pattern avoiding permutations ($r = 0$)

The $r = 0$ case (classical pattern avoidance) has received a lot of attention.

This gets difficult very quickly.

Even for length 4 patterns, many unresolved questions remain (e.g., pattern 1324).

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FYI: There's now a bet whether this will be found by year 2030.

Exactly r copies of a pattern ($r > 0$)

Most work on $r > 0$ focuses on length 3 patterns:

- $s_n(132, r)$ studied by Bóna, Mansour and Vainshtein, Fulmek, and others.
- $s_n(123, r)$ studied by Noonan and Zeilberger, Fulmek, Callan, and others.

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For motivation, we will start with $\tau = 12 \dots k$.

Additional definitions

Assume pattern $\tau = 123$ (equiv. abc).

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Observe: coeff. of t^r in $P_n(t; 1, \ldots, 1) = s_n(123, r)$.

Functional equations for 123

Theorem (Noonan and Zeilberger, 1996)

$$P_n(t;\ x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i^{n-i} P_{n-1}(t;\ x_1,\ldots,x_{i-1},tx_{i+1},\ldots,tx_n)$$

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For example, if $x_1 = \ldots = x_n = 1$, then

$$P_n(t;\ 1\ [n\ \text{times}]) = \sum_{i=1}^{n} P_{n-1}(t;\ 1\ [i-1\ \text{times}], t\ [n-i\ \text{times}])$$

Maple implementation

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Example

- For $r = 0$, the first 10 terms of $s_n(123, r)$ are: $1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$
- For $r = 1$, the first 10 terms of $s_n(123, r)$ are: $0, 0, 1, 6, 27, 110, 429, 1638, 6188, 23256$
- For $r = 6$, the values of $s_n(123, r)$ for $15 \leq n \leq 20$ are: $327200581, 1501719377, 6773007550, 30100185693, 132099138291, 573518305776$

Extending the approach to $\tau = 1234$

Definition

For variables $t, x_1, \ldots, x_n, y_1, \ldots, y_n$, define

$$weight(\pi) :=$$

$$t^{\#\text{ of } abcd \text{ in } \pi} \cdot \prod_{i=1}^{n} x_i^{\#\text{ of } ab \text{ in } \pi \text{ s.t. } a=i} \cdot \prod_{j=1}^{n} y_j^{\#\text{ of } abc \text{ in } \pi \text{ s.t. } a=j}$$

$$P_n(t;\ x_1, \ldots, x_n;\ y_1, \ldots, y_n) := \sum_{\pi \in \mathcal{S}_n} weight(\pi)$$

Observe: coeff. of t^r in $P_n(t;\ 1, \ldots, 1;\ 1, \ldots, 1) = s_n(1234, r)$.

Functional equations for 1234

Theorem (Noonan and Zeilberger, 1996)

$$P_n(t;\ x_1,\ldots,x_n;\ y_1,\ldots,y_n) =$$

$$\sum_{i=1}^{n} x_i^{n-i} P_{n-1}(t;\ x_1,\ldots,x_{i-1}, y_i x_{i+1},\ldots, y_i x_n;\ y_1,\ldots,y_{i-1}, t y_{i+1},\ldots, t y_n)$$

We can use this functional equation to compute $P_n(t;\ 1,\ldots,1;\ 1,\ldots,1)$.

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This approach extendable to any increasing pattern $12\ldots k$.

Further extensions

Previous approach is extendable to 132, 1243, and more generally $12\ldots(k-2)(k)(k-1)$.

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Some additional patterns that can be handled with this approach:

- Patterns 231, 2341, and more generally $23\ldots k1$
  (Recall Vatter's talk and the comment on *adcb*)

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- Patterns 231, 2341, and more generally $23\ldots k1$
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- Pattern 1324

Set-up for 1324

We consider the catalytic variables:

- $x_{i,j}$ $(1 \leq i \leq j \leq n)$
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Variables $x_{i,j}$ will be written as a matrix of variables:

$$X_n := \begin{bmatrix} x_{1,1} & & \cdots & & x_{1,n} \\ & \ddots & & & \\ \vdots & & x_{i,i} & & \vdots \\ & & & \ddots & \\ x_{n,1} & & \cdots & & x_{n,n} \end{bmatrix}$$

(similarly for variables $y_{i,j}$ and matrix Y_n)

Weight function for 1324

Definition

For variables t, $x_{i,j}$'s, and $y_{i,j}$'s $(1 \leq i, j \leq n)$, define

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Definition

$$P_n(t; X_n; Y_n) := \sum_{\pi \in \mathcal{S}_n} weight(\pi)$$

Observe: coeff. of t^r in $P_n(t; \mathbf{1}_n;\ \mathbf{1}_n) = s_n(1324, r)$.

A matrix operator

Define operator R_1 on an arbitrary $n \times n$ square matrix Y_n and $1 \leq i < n$:

$$R_1(Y_n, i) := \begin{bmatrix} y_{1,1} & \cdots & y_{1,i-1} & u_1 & y_{1,i+2} & \cdots & y_{1,n} \\ \vdots & \ddots & & \vdots & & & \vdots \\ y_{i-1,1} & \cdots & y_{i-1,i-1} & u_{i-1} & y_{i-1,i+2} & \cdots & y_{i-1,n} \\ y_{i+1,1} & \cdots & y_{i+1,i-1} & u_{i+1} & y_{i+1,i+2} & \cdots & y_{i+1,n} \\ \vdots & & & \vdots & & \ddots & \vdots \\ y_{n,1} & \cdots & y_{n,i-1} & u_n & y_{n,i+2} & \cdots & y_{n,n} \end{bmatrix}$$

with

$$u_k := t \cdot y_{k,i} \cdot y_{k,i+1}$$

And $R_1(Y_n, n)$ defined to be Y_n with n-th row and n-th column deleted.

... and another

Define operator R_2 on two $n \times n$ square matrices X_n and Y_n and $1 < i \leq n$:

$$R_2(X_n, Y_n, i) := \begin{bmatrix} x_{1,1} & \cdots & x_{1,i-2} & w_1 & x_{1,i+1} & \cdots & x_{1,n} \\ \vdots & \ddots & & \vdots & & & \vdots \\ x_{i-1,1} & \cdots & x_{i-1,i-2} & w_{i-1} & x_{i-1,i+1} & \cdots & x_{i-1,n} \\ x_{i+1,1} & \cdots & x_{i+1,i-2} & w_{i+1} & x_{i+1,i+1} & \cdots & x_{i+1,n} \\ \vdots & & & \vdots & & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,i-2} & w_n & x_{n,i+1} & \cdots & x_{n,n} \end{bmatrix}$$

with

$$w_k := y_{i,1}y_{i,2}\cdots y_{i,k} \cdot x_{k,i-1} \cdot x_{k,i}$$

And $R_2(X_n, Y_n, 1)$ is X_n with the 1-st row and 1-st column deleted.

Functional equation for 1324

Theorem

$$P_n(t; X_n, Y_n) =$$

$$\sum_{i=1}^{n} x_{i,i}^{n-i} x_{i,i+1}^{n-i-1} \cdots x_{i,n-1}^{1} \cdot P_{n-1}(t;\ R_2(X_n, Y_n, i);\ R_1(Y_n, i)).$$

And the coeff. of t^r in $P_n(t; \mathbf{1}_n, \mathbf{1}_n)$ is exactly $s_n(1324, r)$.

Avoiding 1324

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Easy to refine by the number of inversions.
(e.g., confirmed CJS conjecture on 1324-avoiders with exactly k inversions up to $n, k \leq 23$)

Some other recent work

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They conjecture:

$$s_n(1324) \sim B \cdot \mu^n \cdot \mu_1^{\sqrt{n}} \cdot n^g$$

with $B = 9.5 \pm 1.0$, $\mu = 11.60 \pm 0.01$, $\mu_1 = 0.0398 \pm 0.0010$, and $g = -1.1 \pm 0.2$.

Thank you

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