

An Involution on Involutions

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work in progress

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Definition

A *Standard Young Tableau* is a Ferrers shape with n boxes such that each box contains an entry from $\{1, 2, 3, \dots, n\}$ where the columns and rows have entries appearing in increasing order.

Example

1	2	3	6	8
4	7	9		
5				

The Robinson-Schensted-Knuth (RSK) Algorithm creates a bijection between permutations and pairs of Standard Young Tableaux (P, Q) where P and Q have the same shape and are such that the number of boxes in each tableau is the same as the length of the given permutation.

Recall that this algorithm proceeds as follows:

- ① Place the left most entry of the permutation in the top left corner of the P tableau.
- ② For each additional entry k of the permutation, place k in the top row of P by either:
 - placing k at the right end of the row if k is larger than all the current entries in the top row.
 - using k to bump the smallest entry j in the top row such that $k < j$. In this case, repeat this procedure to place j in the second row and to handle all subsequent bumpings.
- ③ As each entry moves into the P tableau, record the order in which the boxes of P are created in the Q tableau.

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \boxed{5}$$

$$Q = \boxed{1}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|} \hline 2 \\ \hline 5 \\ \hline \end{array}$$

$$Q = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|c|c|} \hline 2 & 4 & 8 \\ \hline 5 & & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|c|c|} \hline 2 & 4 & 7 \\ \hline 5 & 8 & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 8 & \\ \hline 5 & & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline 6 & & \\ \hline \end{array}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|c|c|} \hline 1 & 3 & 7 \\ \hline 2 & 4 & \\ \hline 5 & 8 & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline 6 & 7 & \\ \hline \end{array}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 7 & 9 \\ \hline 2 & 4 & & \\ \hline 5 & 8 & & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 8 \\ \hline 2 & 5 & & \\ \hline 6 & 7 & & \\ \hline \end{array}$$

Example

Applying the RSK Algorithm to the permutation $\pi = 524871396$:

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 9 \\ \hline 2 & 4 & 7 & \\ \hline 5 & 8 & & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 8 \\ \hline 2 & 5 & 9 & \\ \hline 6 & 7 & & \\ \hline \end{array}$$

Recall that the inverse of the mapping can be done by using the Q tableau to find where the last box was created and doing a reverse bumping process in P from that box to get the last entry of the permutation.

Theorem

(Schützenberger) For any $\pi \in S_n$, we have $P(\pi) = Q(\pi^{-1})$ and $Q(\pi) = P(\pi^{-1})$.

Corollary

If π is an involution, then $P(\pi) = Q(\pi)$. Thus the RSK algorithm gives a bijection between the set of involutions on $\{1, 2, \dots, n\}$ and the set of Standard Young Tableaux with n boxes.

Let I_n be the set of involutions contained in S_n . Then the bijection we are considering is $f : I_n \rightarrow I_n$ where $f(\pi)$ is obtained by first applying RSK to a given involution π to obtain the associated Standard Young Tableau $P(\pi)$, then taking the transpose of $P(\pi)$, and finally applying the inverse of the RSK Algorithm to $P(\pi)^T$.

Example

Consider the involution $\pi = 5276143$.

1	3
2	4
5	6
7	

Then $P(\pi) =$

7

 and so $P(\pi)^T =$

1	2	5	7
3	4	6	

.

Thus $f(\pi) = 3412657$.

RSK is super duper, but it can be tedious. And then reversing the process means double the work to find f of a given involution.

So is there another way to compute f ?

Answer: There must be. What we currently know how to do only applies for some specific cases though.

One nice case is an immediate consequence of a result of Schensted.

Theorem

For $\pi \in S_n$, we have $P(\pi)^t = P(\pi^r)$.

Thus, in the case when both π, π^r are involutions, we know that $f(\pi) = \pi^r$.

Definition

A tableau Y is said to satisfy the *NE condition* if every entry is larger than its northeast neighbor (one up and one to the right).

We similarly define the *SW condition* if every entry is larger than its southwest neighbor (one below and one to the left).

Example

$P = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array}$
 satisfies the *NE* condition.

Similarly $P^T = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array}$
 satisfies the *SW* condition.

Consider the case where $\pi \in I_n$ is such that 1 and n are in the same 2-cycle. Then we know.

- ① The n enters in the first column and thus n must remain in the first column.
- ② All the other entries are in the SYT before the 1 enters.
- ③ 1 bumps 2 when it enters and then the 2 (and subsequently each bumped entry) bumps the entry below it down one row.
- ④ Since there was a SYT before 1 entered, every entry in column 2 must be larger than its southwest neighbor.

Proposition

Let π be an involution. Then in tableau $P(\pi)$ we have n is alone in the last row and that $P(\pi)$ satisfies the SW condition on the first two columns if and only if 1 and n are in the same 2-cycle in π .

The converse can be seen by simply applying the inverse RSK algorithm to any SYT satisfying the given conditions.

Example

1	3	6
2	5	
4	8	
7		
9		

$$P(978456231) =$$

Building on ideas included in the proposition given above, we can show some properties of f applied to specific sets of involutions. One such result is:

Proposition

If we have that both 1 and n are fixed points, we can apply the map to the remaining portion of the involution knowing that 1 and n will be in a 2-cycle. Basically, $f(1 \oplus \pi \oplus n) = (1n)f(\pi)$ where the entries of $f(\pi)$ are shifted up by 1.

We can also use the *NE* and *SW*-like properties to easily determine how f impacts descents:

Proposition

If π has a descent at i , then $f(\pi)$ has an ascent at i .

Proof.

Recall that a permutation π has a descent at position i if and only if $i + 1$ appears south and weakly west of i in its recording tableau $Q(\pi)$. Taking the transpose of $Q(\pi)$, which is $Q(f(\pi))$, will result in $i + 1$ appearing weakly north (and east) of i . □

Returning to our earlier example:

Example

$$P(\textcolor{red}{1}3265487) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array} .$$

$$\text{and } P^T = P(\textcolor{blue}{7}4825613) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array} .$$

Returning to our example:

Example

$$P(13265487) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array} .$$

$$\text{and } P^T = P(74825613) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array} .$$

Returning to our example:

Example

$$P(13\textcolor{red}{2}65487) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array} .$$

$$\text{and } P^T = P(74\textcolor{blue}{8}25613) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array} .$$

Returning to our example:

Example

$$P(132\textcolor{blue}{6}5487) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array} .$$

$$\text{and } P^T = P(748\textcolor{red}{2}5613) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array} .$$

Returning to our example:

Example

$$P(1326\mathbf{5}487) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array} .$$

$$\text{and } P^T = P(7482\mathbf{5}613) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array} .$$

Returning to our example:

Example

$$P(13265\textcolor{red}{4}87) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array} .$$

$$\text{and } P^T = P(74825\textcolor{blue}{6}13) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array} .$$

Returning to our example:

Example

$$P(132654\textcolor{blue}{8}7) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 8 & \\ \hline 6 & & & \\ \hline \end{array} .$$

$$\text{and } P^T = P(748256\textcolor{red}{1}3) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & 8 & \\ \hline 7 & & \\ \hline \end{array} .$$

Recall Schensted's results on increasing and decreasing sequences.

Theorem

Given $\pi \in S_n$, the length of the first row of $P(\pi)$ is the length of the longest increasing subsequence of π .

Similarly, the length of the first column of $P(\pi)$ is the length of the longest decreasing subsequence of π .

As such, it is more manageable to restrict ourselves to involutions that avoid a monotone sequence of length three.

Definition

Let π be an involution. Classify each entry of π as *fixed*, *small*, or *large* as follows:

- fixed entries are the usual fixed points.
- **small entries are the entries that are the smaller of the two entries in their transposition.**
- **large entries are the entries that are the larger of the two entries in their transposition.**

Example

$$\pi = 351624798 = (13)(25)(46)(7)(89)$$

Proposition

Let π be a 321-avoiding involution. The first row of the Standard Young Tableau of π consists all small entries and all fixed points. The second row therefore consists of all large entries.

One can see that no small entry or fixed point can be bumped out of the first row since π avoids 321. Then all large entries must be bumped as otherwise the length of the first row would force both entries of a 2-cycle of π to be in an increasing sequence in π .

Example

$$\pi = 351624798 = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 6 & 9 & \\ \hline \end{array}$$

Now consider τ to be a 123-avoiding involution. We will label each entry τ_i of τ by the length of the longest decreasing subsequence of τ that terminates at τ_i . As τ avoids 123, any label can be shared by at most two entries.

Definition

We will define τ_i to be a *record breaker* to be an entry of τ that is the first entry (from left to right) to have its label. That is, every maximum length descending subsequence of $\tau_1\tau_2\ldots\tau_i$ must include τ_i .

Example

$\tau = 849276513$

Notice that anytime a record breaker τ_i enters the P tableau for τ , the length of the first column must increase by one. Also, because $P(\tau) = Q(\tau)$, we know that i must be in the first column of $P(\tau)$. Similarly, if τ_j is not a record breaker, we know that j must be in the second column of $P(\tau)$.

Example

Recall that τ has record breakers in positions 1, 2, 4, 7, 8.

1	3
2	5
4	6
7	9
8	

$\tau = 849276513 =$

Using this information, we can compute $f(\tau)$ without actually applying the RSK algorithm.

- 1 Determine which entries of τ are record breakers.
- 2 Consider τ_n . If τ_n is a record breaker, then n is a fixed point in $f(\tau)$. Otherwise n is in a 2-cycle with the largest (remaining) index of a record breaker.
- 3 Continue this process from right to left.

Example

$$\begin{aligned}
 f(\tau) &= f(849276513) \\
 &= f(8492765)(98) \\
 &= f(849276)(7)(98) \\
 &= f(849\Box 7)(46)(7)(98) \\
 &= f(8\Box 9)(25)(46)(7)(98) \\
 &= (13)(25)(46)(7)(98)
 \end{aligned}$$

Thank you!