

Open problem: 4231-avoiding minimal depth permutations

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based on discussions with Kyle Petersen (DePaul)

Permutation Patterns, Johnson City, TN
July 9, 2014

Depth

Kyle Petersen and Bridget Tenner (arXiv: 1202.4765v2) define the **depth** of a Coxeter group element. Their primary definition is Coxeter theoretic, but they prove the following for permutations:

$$\text{dp}(w) = \sum_{w(i) > i} (w(i) - i) = \frac{1}{2} \sum_{i=1}^n |w(i) - i|$$

(which is a statistic previously studied by Diaconis and Graham.)

Length and reflection length

The **length** of a permutation $\ell(w)$ is the number of inversions - pairs $i < j$ with $w(i) > w(j)$. (This is the length of the shortest way to write w as a product of adjacent transpositions.)

The **reflection length** of a permutation $\ell'(w)$ is $n - \text{cyc}(w)$, where $\text{cyc}(w)$ is the number of disjoint cycles. (This is the length of the shortest way to write w as a product of transpositions.)

Petersen and Tenner observe (trivially from their definition, previously given a complicated proof by Diaconis and Graham) that

$$\text{dp}(w) \geq \frac{\ell(w) + \ell'(w)}{2}.$$

We'll call the permutations where equality holds **permutations of minimal depth**.

Permutations of minimal depth

Petersen and Tenner point out that the class of permutations of minimal depth is *not* a pattern class.

$v = 3412$ does not have minimal depth ($\text{dp}(v) = 4$, $\ell(v) = 4$, $\ell'(v) = 2$)

$w = 53412$ contains v but does have minimal depth ($\text{dp}(w) = 6$, $\ell(w) = 8$, $\ell'(w) = 4$)

We will call a permutation **buoyant** if it has minimal depth but contains a permutation that does not.

Conjecture

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All buoyant permutations contain 4231.

This conjecture can be rephrased:

Conjecture

The set of permutations that avoid 4231 and have minimal depth is a pattern class.

And I have a more precise version (checked through $\mathfrak{S}_9$):

Conjecture

A 4231-avoiding permutation has minimal depth iff it avoids 4231, 3412, 34512, 54123, 365214, 541632, 7652143, and 5476321.