Pattern avoidance on clans, skew partitions, and pairs of 2-block set partitions

Alexander Woo (U. Idaho)
joint work with Ben Wyser (UIUC/Grenoble))

Permutation Patterns, Johnson City, TN
July 10, 2014

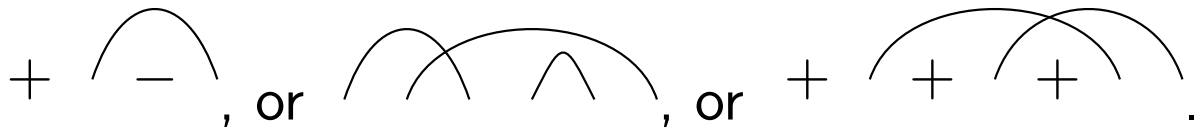
Clans

A **clan** is a partial matching on n linearly ordered vertices, with unmatched vertices given a sign (i.e. $+$ or $-$).

Clans

A **clan** is a partial matching on n linearly ordered vertices, with unmatched vertices given a sign (i.e. $+$ or $-$).

We draw clans like:



Pattern avoidance on clans

There is a natural notion of pattern avoidance on clans:

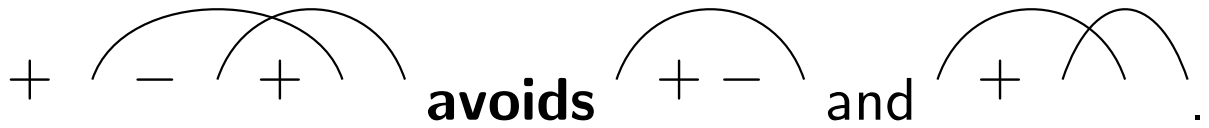
.

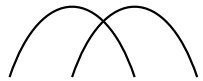
Pattern avoidance on clans

There is a natural notion of pattern avoidance on clans:

Pattern avoidance on clans

There is a natural notion of pattern avoidance on clans:



A **non-crossing** clan is one that avoids .

Clans and Algebraic Geometry (and Rep Thy)

Given positive integers p and q, a (p, q)-clan is a clan where p is the number of matchings plus the number of $+$'s, and q the number of matchings plus the number of $-$'s. (This means $n = p + q$ is the total number of positions.)

Clans and Algebraic Geometry (and Rep Thy)

Given positive integers p and q, a (p, q)-clan is a clan where p is the number of matchings plus the number of $+$'s, and q the number of matchings plus the number of $-$'s. (This means $n = p + q$ is the total number of positions.)

The group $GL_p \times GL_q$ acts on the flag variety GL_n/B, and the orbits of this action are in one-to-one correspondence with (p, q)-clans. The geometry of these orbits and orbit closures give information about the representations of the real Lie group $SU(p, q)$.

Geometry and Pattern avoidance

Let Z_γ be the closure of the orbit associated to a clan γ. When γ is non-crossing, there are permutations u and v in S_n such that $Z_\gamma = X_v^u$, where X_v^u is a **Richardson variety**.

Geometric properties of Richardson varieties are determined by **interval pattern avoidance**. So to understand geometric properties of non-crossing clans, we need to translate interval pattern avoidance conditions into conditions on clans through the implicit bijection between γ and (u, v).

Bruhat order

For a permutation $\sigma \in S_n$, let $\mathrm{inv}(\sigma)$ be the number of **inversions** - pairs $i < j$ with $\sigma(i) > \sigma(j)$.

Bruhat order

For a permutation $\sigma \in S_n$, let $\mathrm{inv}(\sigma)$ be the number of **inversions** - pairs $i < j$ with $\sigma(i) > \sigma(j)$.

Let $\sigma, \tau \in S_n$. We say τ **covers** σ in Bruhat order if:

- There is a transposition t such that $\tau = t\sigma$
- $\mathrm{inv}(\tau) = \mathrm{inv}(\sigma) + 1$.

Bruhat order is the transitive closure of this covering relation.

Bruhat order

For a permutation $\sigma \in S_n$, let $\mathrm{inv}(\sigma)$ be the number of **inversions** - pairs $i < j$ with $\sigma(i) > \sigma(j)$.

Let $\sigma, \tau \in S_n$. We say τ **covers** σ in Bruhat order if:

- There is a transposition t such that $\tau = t\sigma$
- $\mathrm{inv}(\tau) = \mathrm{inv}(\sigma) + 1$.

Bruhat order is the transitive closure of this covering relation.

For example, $45312 > 25413$ since 45312 covers 45213 which covers 25413.

Interval pattern avoidance

Suppose $u, v \in S_m$ and $x, w \in S_n$. We say $[x, w]$ **interval contains** $[u, v]$ if there exist indices $i_1 < \cdots < i_m$ such that:

- w contains v and x contains u at those indices
  $w(i_j) > w(i_k)$ iff $v(j) > v(k)$; $x(i_j) > x(i_k)$ iff $u(j) > u(k)$

Interval pattern avoidance

Suppose $u, v \in S_m$ and $x, w \in S_n$. We say $[x, w]$ **interval contains** $[u, v]$ if there exist indices $i_1 < \cdots < i_m$ such that:

- w contains v and x contains u at those indices
  $w(i_j) > w(i_k)$ iff $v(j) > v(k)$; $x(i_j) > x(i_k)$ iff $u(j) > u(k)$
- w and x agree outside of those indices
  $w(a) = x(a)$ if $a \neq i_j$ for all j, $1 \leq j \leq m$.

Interval pattern avoidance

Suppose $u, v \in S_m$ and $x, w \in S_n$. We say $[x, w]$ **interval contains** $[u, v]$ if there exist indices $i_1 < \cdots < i_m$ such that:

- w contains v and x contains u at those indices
  $w(i_j) > w(i_k)$ iff $v(j) > v(k)$; $x(i_j) > x(i_k)$ iff $u(j) > u(k)$
- w and x agree outside of those indices
  $w(a) = x(a)$ if $a \neq i_j$ for all j, $1 \leq j \leq m$.
- $\mathrm{inv}(w) - \mathrm{inv}(x) = \mathrm{inv}(v) - \mathrm{inv}(u)$

We say $[x, w]$ **interval avoids** $[u, v]$ if it does not interval contain it.

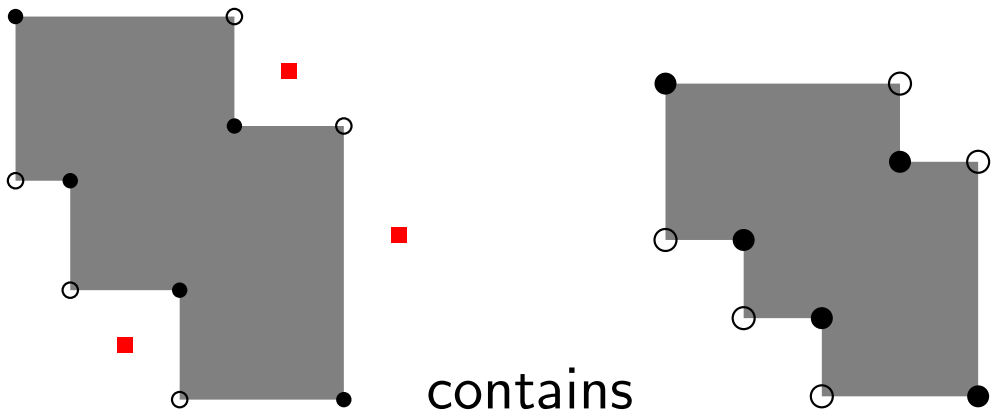
Pictures for interval pattern avoidance

When $u \leq v$ (Bruhat order), then interval pattern avoidance is a form of mesh avoidance. The forbidden regions are those that produce more inversions in v than in u. (One of the alternative definitions of Bruhat order says there are no regions that produce more inversions in u than in v.)

Pictures for interval pattern avoidance

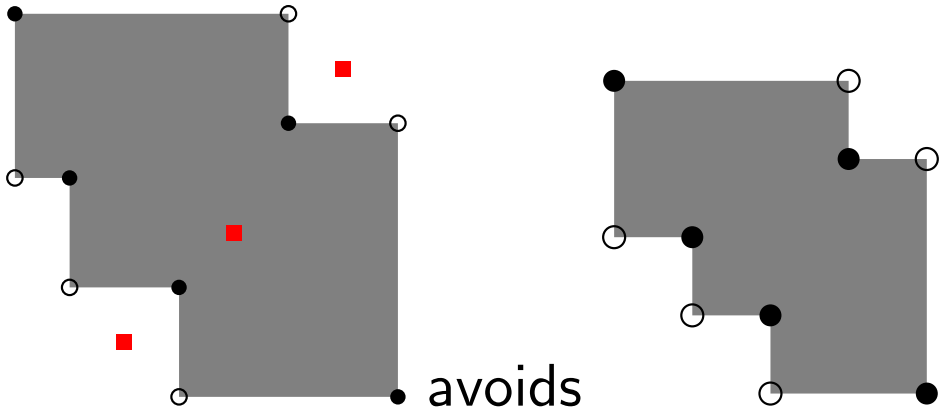
When $u \leq v$ (Bruhat order), then interval pattern avoidance is a form of mesh avoidance. The forbidden regions are those that produce more inversions in v than in u. (One of the alternative definitions of Bruhat order says there are no regions that produce more inversions in u than in v.)

For example, $[53218764, 85236714]$ contains $[32154, 53241]$



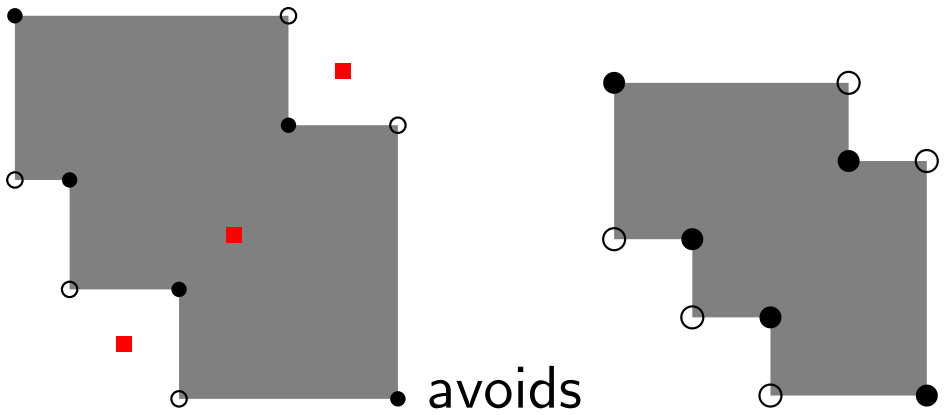
Pictures for interval pattern avoidance, II

However, $[53214876, 85234671]$ avoids $[32154, 53241]$



Pictures for interval pattern avoidance, II

However, $[53214876, 85234671]$ avoids $[32154, 53241]$



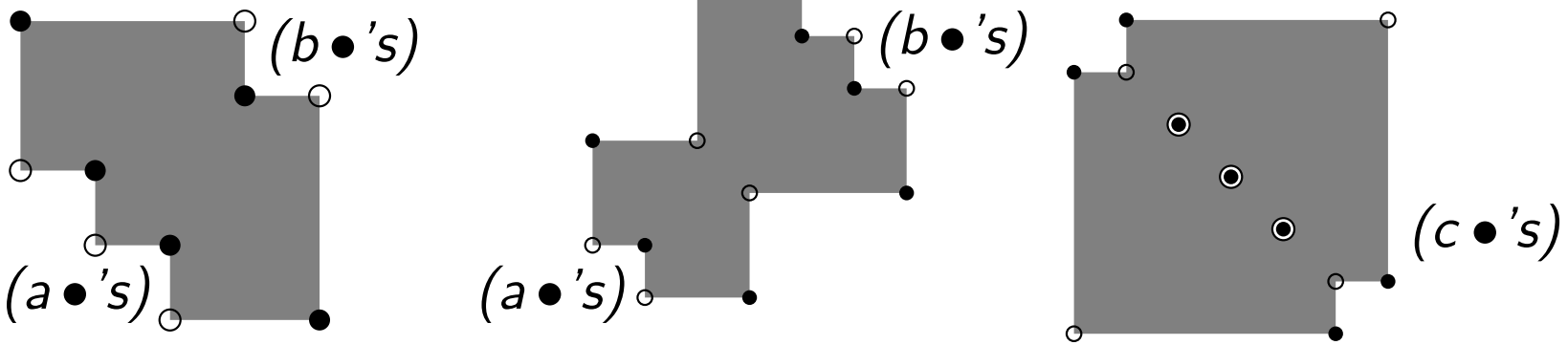
This doesn't seem very interesting, because the pair of permutations $[x, w]$ already tells you where $[u, v]$ has to be contained. What will happen is that x will be allowed to vary (over all permutations of the same size, or some more restricted set).

Smoothness by interval pattern avoidance

Our combinatorial starting point is the following theorem (combining Kuntson-W-Yong with Billey–Warington, Cortez, Kassel–Lascoux-Reutenauer, Manivel):

Theorem

The Richardson variety X_τ^σ is smooth if and only if, for all ρ with $\tau \leq \rho \leq \sigma$, both $[\rho, \sigma]$ and $[\rho^c, \tau^c]$ avoid all of the following:



Cosets and 2-block set partitions

Given a permutation $v \in S_n$ and an index p, we can associate to v the 2-block set partition

$$\mathbf{v} = \{v(1), \ldots, v(p)\} \mid \{v(p+1), \ldots, v(n)\}.$$

The 2-block set partition $\mathbf{v}$ labels the coset of $v(S_p \times S_{n-p})$.

Cosets and 2-block set partitions

Given a permutation $v \in S_n$ and an index p, we can associate to v the 2-block set partition

$$\mathbf{v} = \{v(1), \ldots, v(p)\} \mid \{v(p+1), \ldots, v(n)\}.$$

The 2-block set partition $\mathbf{v}$ labels the coset of $v(S_p \times S_{n-p})$.

An **ascent** of a permutation σ is an index i such that $\sigma(i) < \sigma(i+1)$.

A 2-block set partition $\mathbf{v}$ is associated to a unique Bruhat-maximal permutation v. List the first block in decreasing order, then the second block in decreasing order. This permutation has at most one ascent, always at p.

2-block set partitions and integer partitions

There is a canonical bijection between 2-block set partitions and integer partitions:

- Put down your pencil tip somewhere.

2-block set partitions and integer partitions

There is a canonical bijection between 2-block set partitions and integer partitions:

- Put down your pencil tip somewhere.
- Without picking up your pencil, at the i-th step,

2-block set partitions and integer partitions

There is a canonical bijection between 2-block set partitions and integer partitions:

- Put down your pencil tip somewhere.
- Without picking up your pencil, at the i-th step,
  - draw a segment down if i is in the first block

2-block set partitions and integer partitions

There is a canonical bijection between 2-block set partitions and integer partitions:

- Put down your pencil tip somewhere.
- Without picking up your pencil, at the i-th step,
  - draw a segment down if i is in the first block
  - draw a segment to the right if i is in the second block

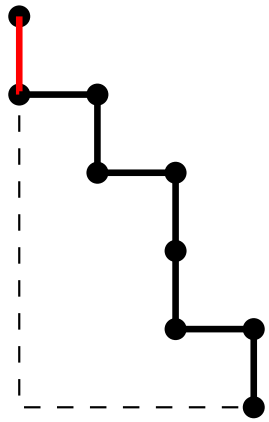
2-block set partitions and integer partitions

There is a canonical bijection between 2-block set partitions and integer partitions:

- Put down your pencil tip somewhere.
- Without picking up your pencil, at the i-th step,
  - draw a segment down if i is in the first block
  - draw a segment to the right if i is in the second block
- You have drawn the outer boundary of your French partition diagram.

2-block set partitions and integer partitions, example

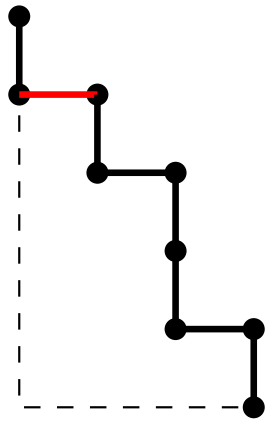
For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

2-block set partitions and integer partitions, example

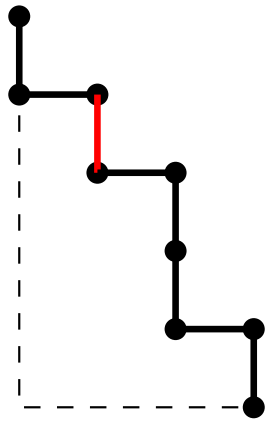
For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

2-block set partitions and integer partitions, example

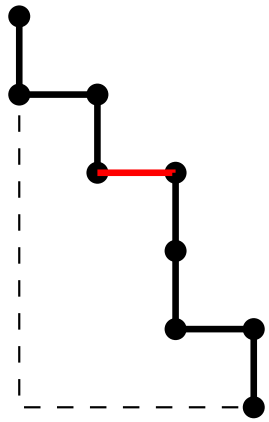
For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

2-block set partitions and integer partitions, example

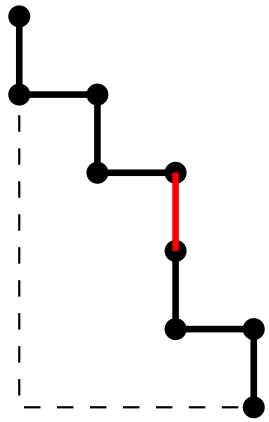
For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

2-block set partitions and integer partitions, example

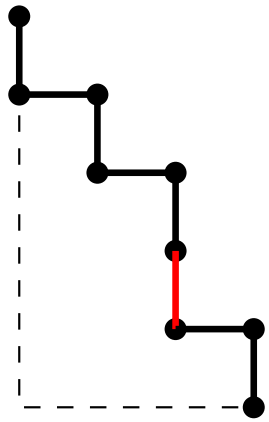
For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

2-block set partitions and integer partitions, example

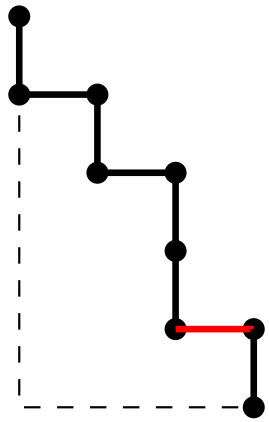
For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

2-block set partitions and integer partitions, example

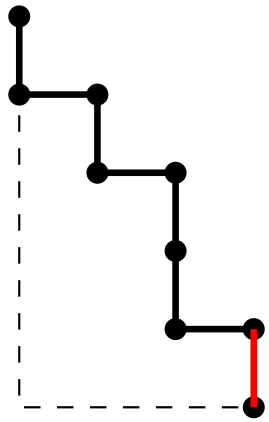
For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

2-block set partitions and integer partitions, example

For example, for $13568|247$, we draw:



which is the partition $(3, 2, 2, 1, 0)$. (For each number in the left block, we list the number of entries smaller than it that are in the right block.)

Pairs of 2-block set partitions and integer partitions

Bruhat order translates to containment of partitions - if v has no ascents except at p, $x \leq v$ iff $\lambda(\mathbf{x}) \subseteq \lambda(\mathbf{v})$.

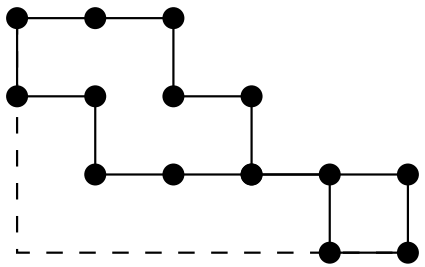
This means a pair of 2-block set partitions $\mathbf{u} \leq \mathbf{v}$ translates to a skew partition.

Pairs of 2-block set partitions and integer partitions

Bruhat order translates to containment of partitions - if v has no ascents except at p, $x \leq v$ iff $\lambda(\mathbf{x}) \subseteq \lambda(\mathbf{v})$.

This means a pair of 2-block set partitions $\mathbf{u} \leq \mathbf{v}$ translates to a skew partition.

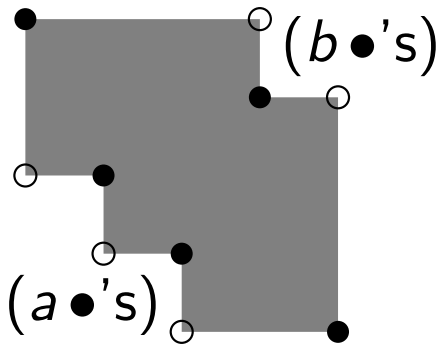
For example, $137|24568 < 358|12467$ gives the picture



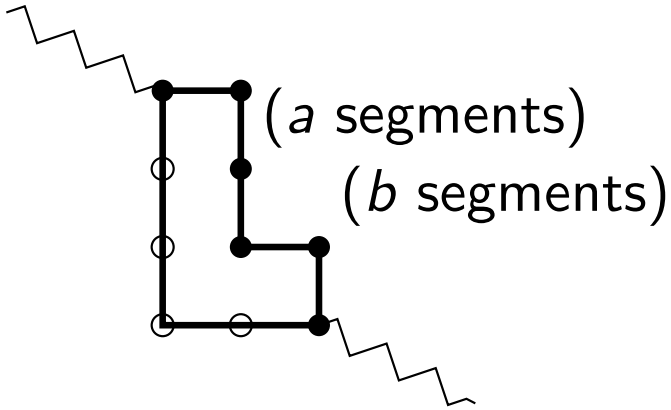
which is the skew partition$(5,3,2)/(4,1,0)$.

Interval pattern conditions for skew partitions

When v has at most one ascent and u at most one descent, both at p, only the first type of interval pattern for smoothness is relevant. In terms of partitions $[x, v]$ containing a pattern of the first type means:

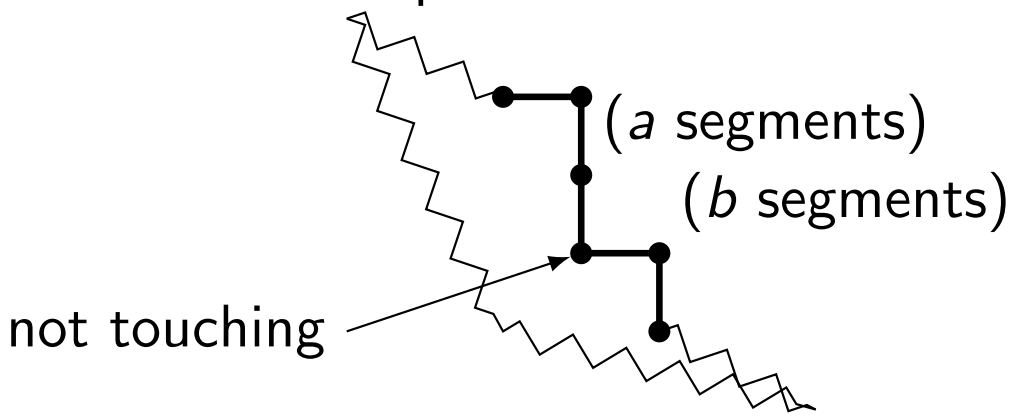


which translates to



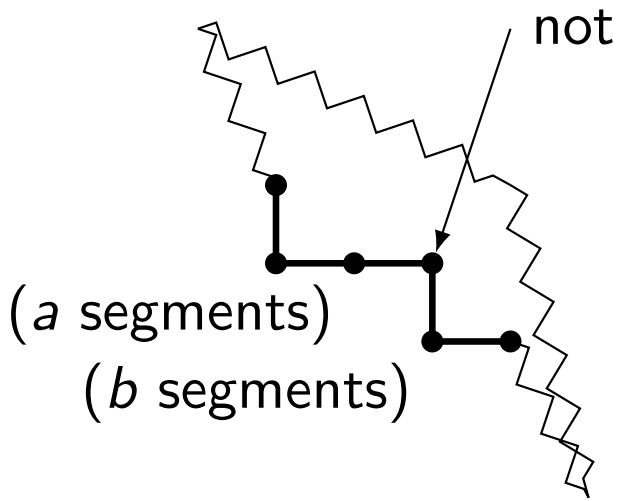
Smoothness in terms of skew partitions

Given a pair $u \leq v$, there exists some x such that $u \leq x \leq v$ and $[x, v]$ contains a pattern of the first type iff the boundary for u is below what is required for x:



Complement for skew partitions

Taking the complement flips a partition along the NW-SE diagonal, so the $[x^c, u^c]$ contains a pattern of the first type for some x with $u \leq x \leq v$ if the skew partition for $[u, v]$ looks like



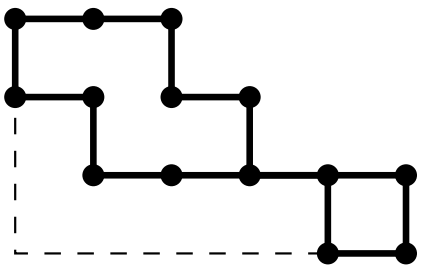
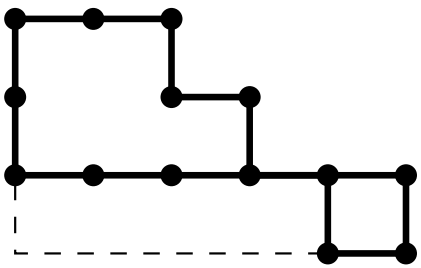
Smoothness for skew partitions

Assume u has at most one descent at p, and v has at most one ascent, also at p. We just saw that X_u^v is smooth if and only if none of the "components" of $\lambda(\mathbf{v})/\lambda(\mathbf{u})$ has an **internal inner corner**.

Smoothness for skew partitions

Assume u has at most one descent at p, and v has at most one ascent, also at p. We just saw that X_u^v is smooth if and only if none of the "components" of $\lambda(\mathbf{v})/\lambda(\mathbf{u})$ has an **internal inner corner**.

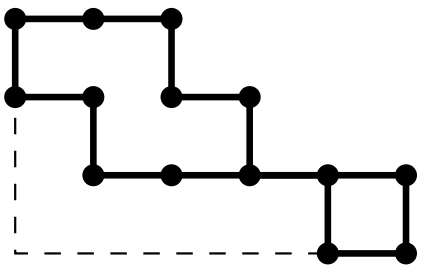
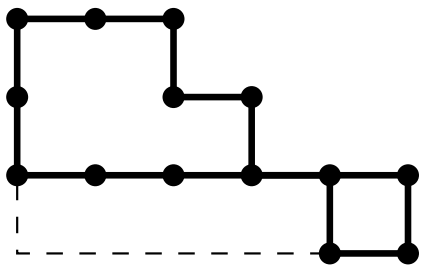
For example,

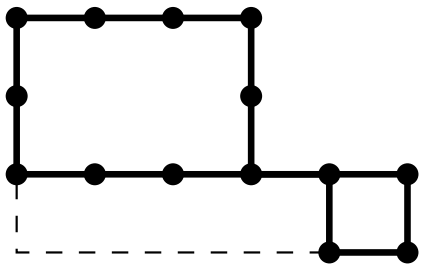
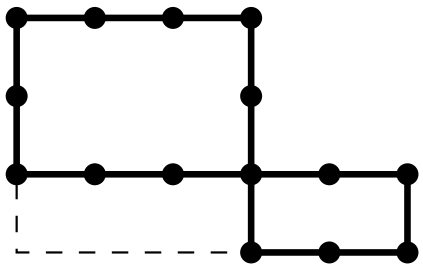
 and  are not smooth.

Smoothness for skew partitions

Assume u has at most one descent at p, and v has at most one ascent, also at p. We just saw that X_u^v is smooth if and only if none of the "components" of $\lambda(\mathbf{v})/\lambda(\mathbf{u})$ has an **internal inner corner**.

For example,


and

are not smooth.


and

are smooth.

Clans to skew partitions

The implicit bijection between a noncrossing clan γ and a pair of permutations (u, v) (so that $Z_\gamma = X_u^v$) can be explained in terms of the skew partition for u and v:
At the i-th step:

Clans to skew partitions

The implicit bijection between a noncrossing clan γ and a pair of permutations (u, v) (so that $Z_\gamma = X_u^v$) can be explained in terms of the skew partition for u and v:
At the i-th step:

- If the i-th entry of γ starts a matching, the paths go away from each other.

Clans to skew partitions

The implicit bijection between a noncrossing clan γ and a pair of permutations (u, v) (so that $Z_\gamma = X_u^v$) can be explained in terms of the skew partition for u and v:
At the i-th step:

- If the i-th entry of γ starts a matching, the paths go away from each other.
- If the i-th entry of γ is a $+$, the paths go down.

Clans to skew partitions

The implicit bijection between a noncrossing clan γ and a pair of permutations (u, v) (so that $Z_\gamma = X_u^v$) can be explained in terms of the skew partition for u and v:
At the i-th step:

- If the i-th entry of γ starts a matching, the paths go away from each other.
- If the i-th entry of γ is a $+$, the paths go down.
- If the i-th entry of γ is a $-$, the paths go right.

Clans to skew partitions

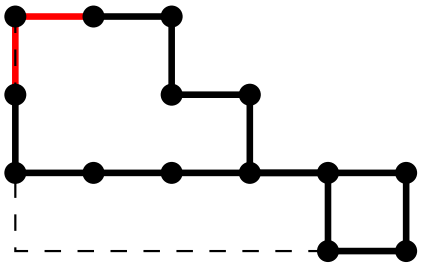
The implicit bijection between a noncrossing clan γ and a pair of permutations (u, v) (so that $Z_\gamma = X_u^v$) can be explained in terms of the skew partition for u and v:
At the i-th step:

- If the i-th entry of γ starts a matching, the paths go away from each other.
- If the i-th entry of γ is a $+$, the paths go down.
- If the i-th entry of γ is a $-$, the paths go right.
- If the i-th entry of γ ends a matching, the paths go towards each other.

Clan bijection example

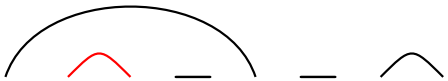
For example,

corresponds to

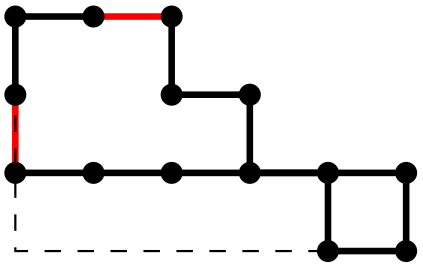


Clan bijection example

For example,

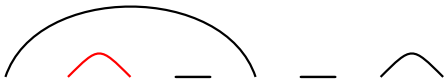


corresponds to

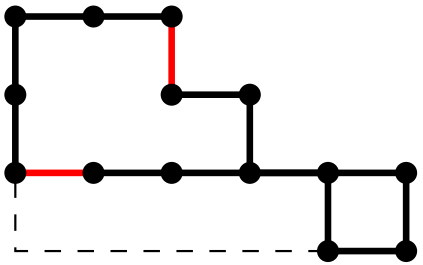


Clan bijection example

For example,



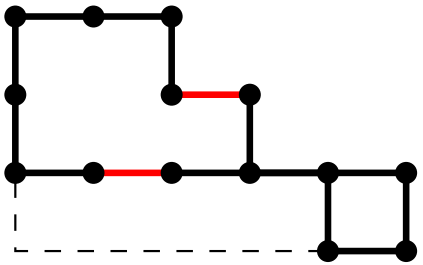
corresponds to



Clan bijection example

For example,

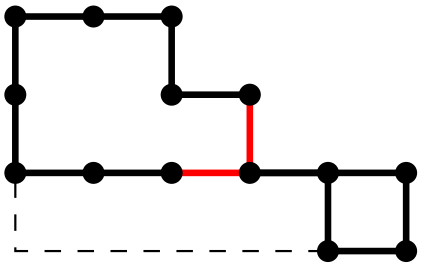
corresponds to



Clan bijection example

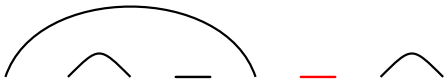
For example,

corresponds to

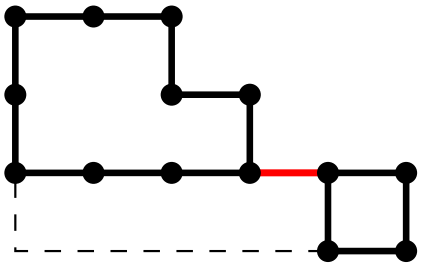


Clan bijection example

For example,

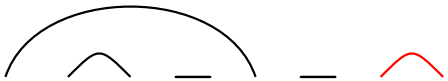


corresponds to

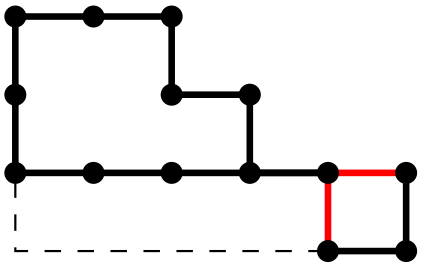


Clan bijection example

For example,

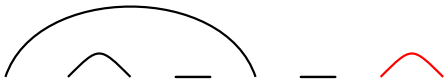


corresponds to

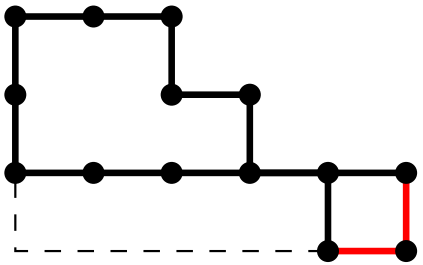


Clan bijection example

For example,

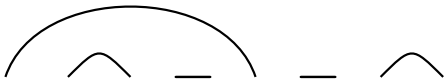


corresponds to

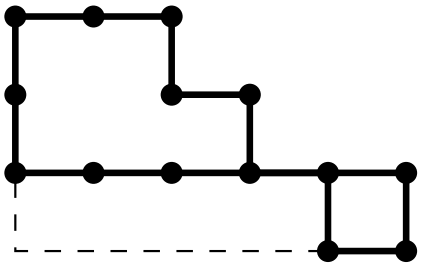


Clan bijection example

For example,



corresponds to

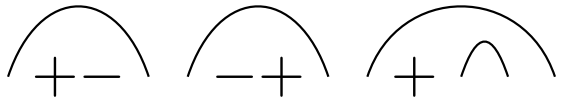


If we want to continue the bijection to permutations, we get $12734568 < 85376421$.

Smoothness for non-crossing clans

Translating avoidance of an internal inner corner to clans gives:

Theorem

Let γ be a non-crossing clan. Then Z_γ is NOT smooth if and only if γ contains one of the following 7 clans: , , *and*

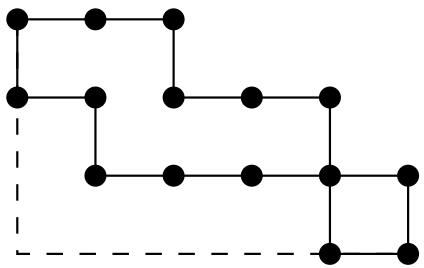
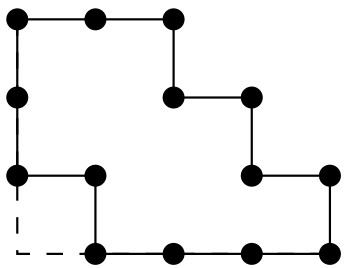
LCI for skew partitions

By a theorem of Darayon, X_u^v is a **local complete intersection** iff each component of the skew partition has at most one internal inner corner on each side, and the internal inner corners have $a = b = 1$.

LCI for skew partitions

By a theorem of Darayon, X_u^v is a **local complete intersection** iff each component of the skew partition has at most one internal inner corner on each side, and the internal inner corners have $a = b = 1$.

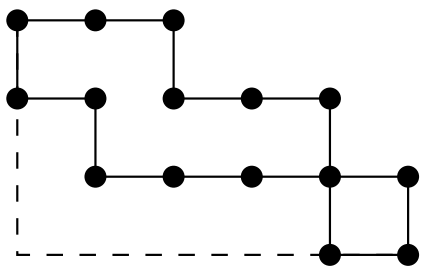
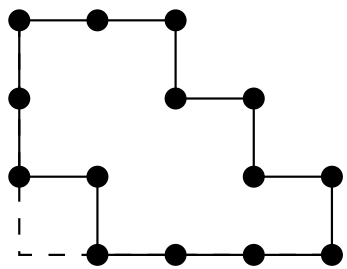
For example,

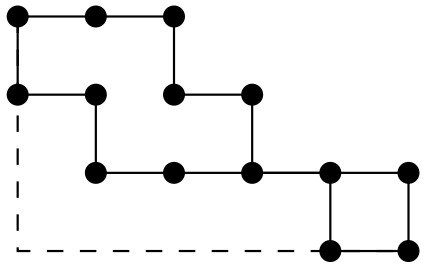
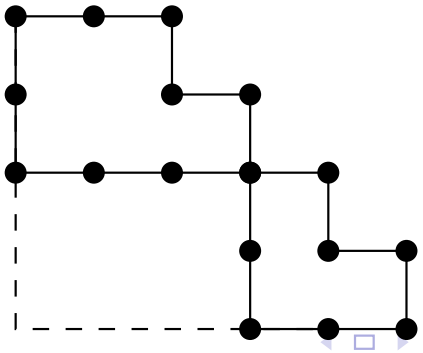
 and  are not lci.

LCI for skew partitions

By a theorem of Darayon, X_u^v is a **local complete intersection** iff each component of the skew partition has at most one internal inner corner on each side, and the internal inner corners have $a = b = 1$.

For example,

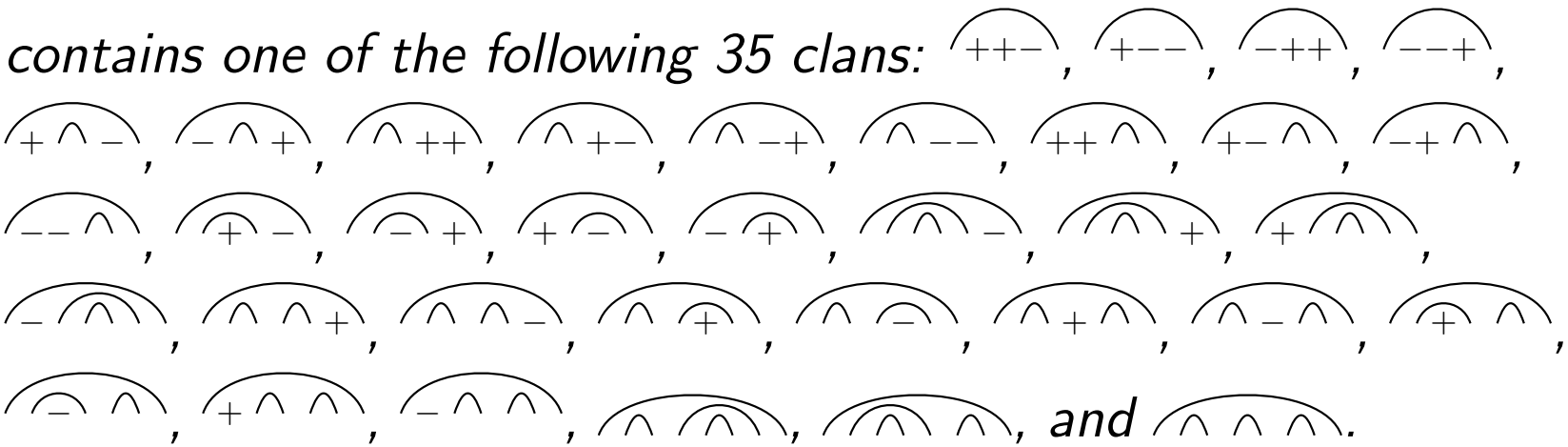

and

are not lci.


and

are lci.

LCI in terms of non-crossing clans

Translating this into clans gives:

Theorem

Let γ be a non-crossing clan. Then Z_γ is NOT lci if and only if γ contains one of the following 35 clans: 

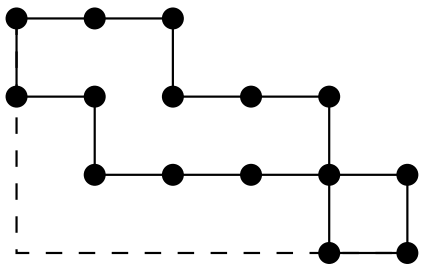
Gorenstein for skew partitions

In terms of the skew partition, X_u^v is a **Gorenstein** if and only if all the internal inner corners have $a = b$. (Perrin)

Gorenstein for skew partitions

In terms of the skew partition, X_u^v is a **Gorenstein** if and only if all the internal inner corners have $a = b$. (Perrin)

For example,

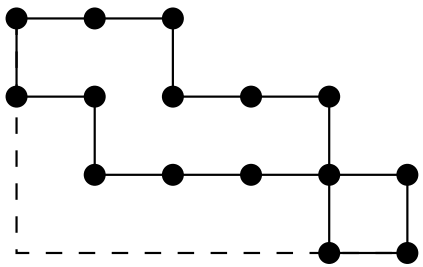


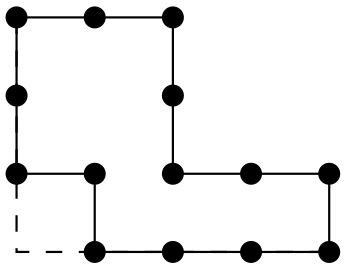
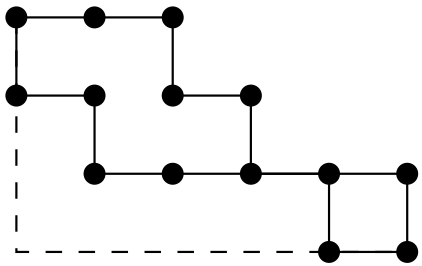
is not Gorenstein.

Gorenstein for skew partitions

In terms of the skew partition, X_u^v is a **Gorenstein** if and only if all the internal inner corners have $a = b$. (Perrin)

For example,


is not Gorenstein.


and

are Gorenstein.

This does not translate to an ordinary pattern avoidance condition.

Conclusion

- There is a geometrically motivated family of combinatorial objects called **clans** with a natural notion of pattern avoidance.
- There are bijections between non-crossing clans, skew partitions, 2-block set partitions, and pairs of permutations with a unique ascent/descent.
- (Interval) pattern conditions can be translated across these bijections.

Conclusion

- There is a geometrically motivated family of combinatorial objects called **clans** with a natural notion of pattern avoidance.
- There are bijections between non-crossing clans, skew partitions, 2-block set partitions, and pairs of permutations with a unique ascent/descent.
- (Interval) pattern conditions can be translated across these bijections.

Thank you for your attention.