



The Open University

A fast-growing subset of $Av(1324)$

David Bevan

Permutation Patterns 2014, East Tennessee State University

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Permutations



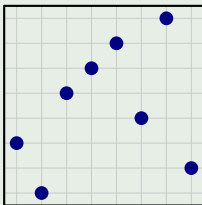
- Permutation of length n : an ordering on $1, \dots, n$.

Example

$$\sigma = 31567482$$

- Graphical perspective: plot the points (i, σ_i) .
- One point in each row; one point in each column.

Example (31567482)

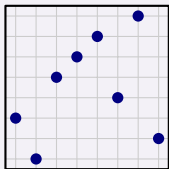


Containment and avoidance



Containment

σ **contains** π if some subsequence of σ has the same ordering as π .



contains

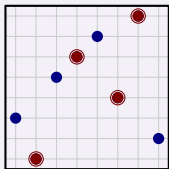


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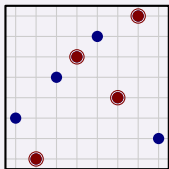


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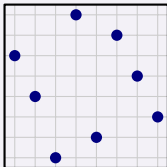


contains



Avoidance

Otherwise σ **avoids** π .



avoids



Permutation classes and growth rates



Definition (permutation class)

$\mathbf{Av}(\pi)$: *class of permutations avoiding π .*

Definition (growth rate)

Growth rate of $\mathbf{Av}(\pi)$:

$$\text{gr}(\mathbf{Av}(\pi)) = \lim_{n \rightarrow \infty} \sqrt[n]{|\mathbf{Av}_n(\pi)|},$$

where $|\mathbf{Av}_n(\pi)|$ = *number of elements of $\mathbf{Av}(\pi)$ of length n .*

$Av(1324)$: What we know



- Notorious for its difficulty; asymptotic growth rate not known.

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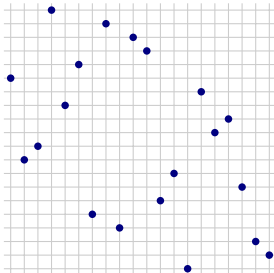


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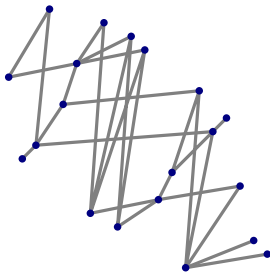
The new result:

- Lower bound of 9.81 on growth rate.

Permutations as Hasse graphs

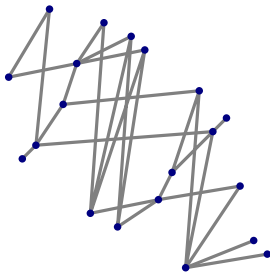


Permutations as Hasse graphs



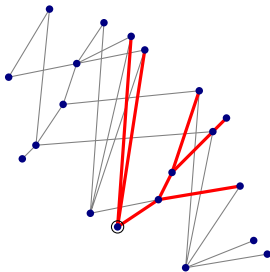
- Ordered graph.
- Hasse diagram of poset on points, where $(i, \sigma_i) < (j, \sigma_j)$ iff $i < j$ and $\sigma_i < \sigma_j$.
- Minima in poset are left-to-right minima of permutation; maxima are right-to-left maxima of permutation.

Hasse graphs of 1324-avoiders



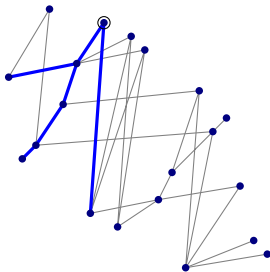
- Avoids  as a minor.
- Up-sets and down-sets are trees.

Hasse graphs of 1324-avoiders



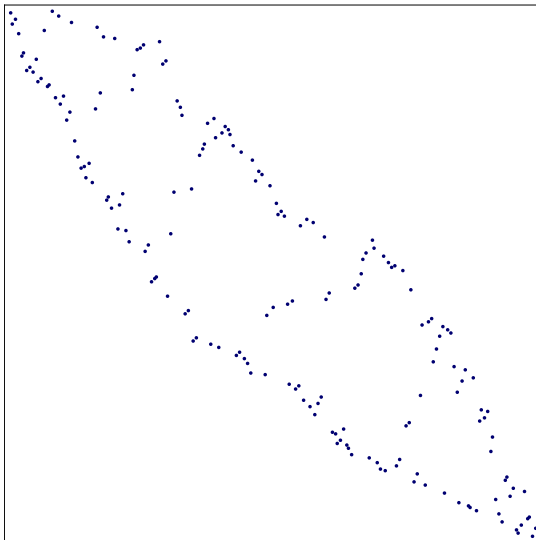
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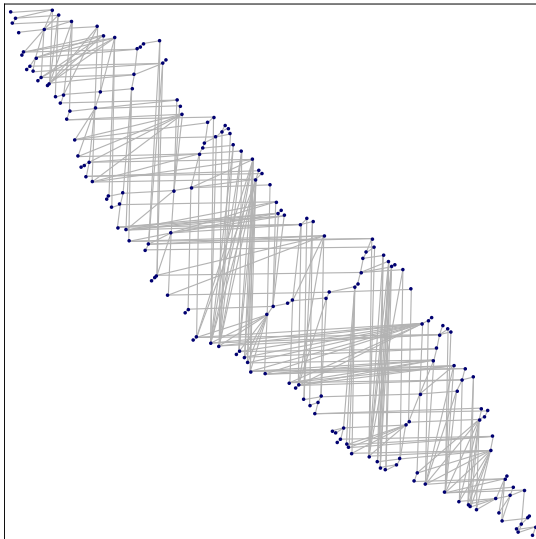


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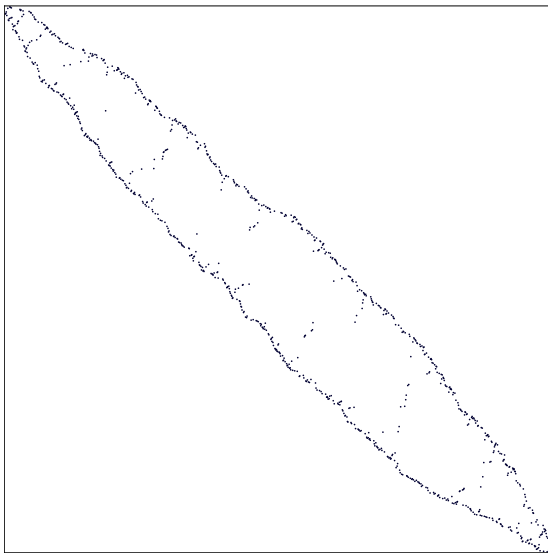
A typical 1324-avoider of length 187



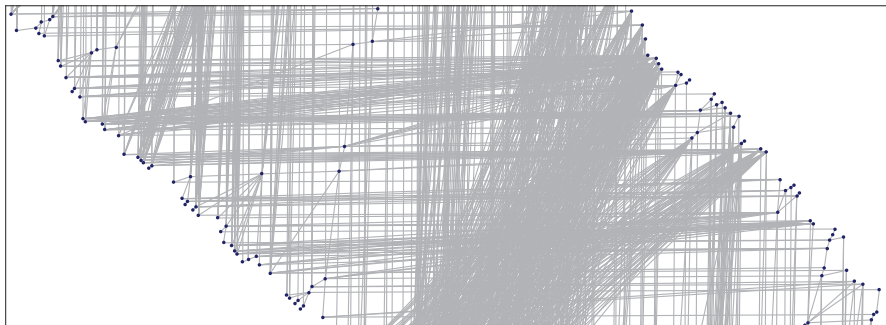
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A typical 1324-avoider of length 1000



Structure of a typical 1324-avoider



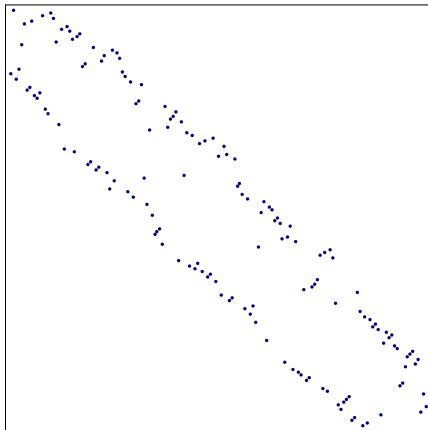
- Cigar-shaped boundary regions consisting of numerous small subtrees.
- Scarcity of points in the interior, partitioned into a few paths connecting the two boundaries.
- Much still a mystery.

1324-avoiders with a regular structure



Definition ($\mathcal{W}$)

*1324-avoiders whose Hasse graphs are spanned by a sequence of trees rooted alternately at the lower left (**red**) and the upper right (**blue**).*

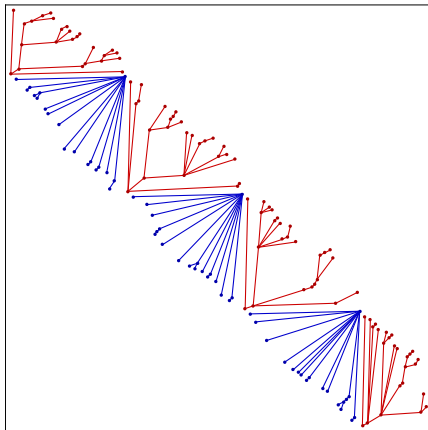


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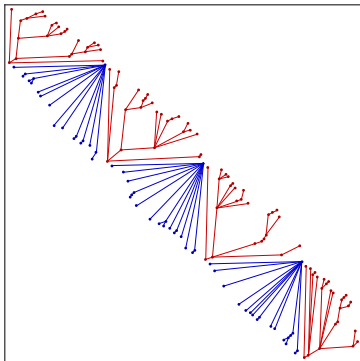


1324-avoiders with a very regular structure



$\mathcal{W}(t, k, \ell, d)$

- There are t blue trees and $t + 1$ red trees.
- Each red tree has k vertices.
- Each blue tree has ℓ vertices.
- Each blue tree has root degree d (d principal subtrees).



$\in \mathcal{W}(3, 25, 19, 12)$

$$\mathcal{W}_{\lambda,\delta}(k)$$



Number and sizes of trees grow together, along with root degree of blue trees.

Given $\lambda > 0$ and $\delta \in (0, 1)$:

$$\mathcal{W}_{\lambda,\delta}(k) = \mathcal{W}(k, k, \lceil \lambda k \rceil, \lceil \delta \lambda k \rceil)$$

- k blue trees and $k + 1$ red trees.
- Each red tree has k vertices.
- Each blue tree has $\ell = \lceil \lambda k \rceil$ vertices.
- Each blue tree has root degree $d = \lceil \delta \lambda k \rceil$.

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Fix λ and δ ; let $k \rightarrow \infty$.

- λ : asymptotic ratio of size of blue trees to size of red trees.
- δ : asymptotic ratio of root degree of blue trees to their size.



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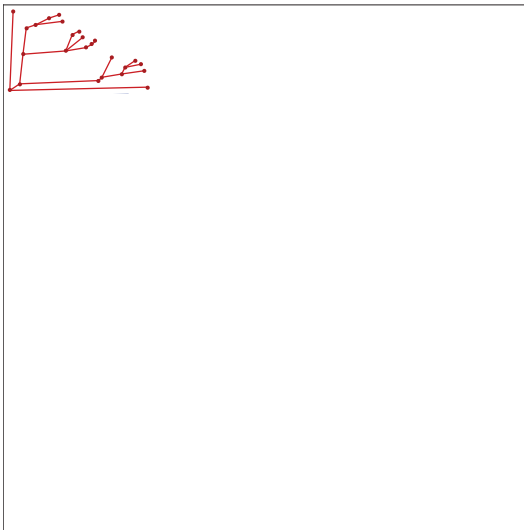
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- λ : asymptotic ratio of size of blue trees to size of red trees.
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- Calculate growth rate $g(\lambda, \delta) = \lim_{k \rightarrow \infty} |\mathcal{W}_{\lambda,\delta}(k)|^{1/n(k,\lambda)}$,
where $n(k, \lambda)$ is length of permutations in $\mathcal{W}_{\lambda,\delta}$.

Building 1324-avoiders in $\mathcal{W}(t, k, \ell, d)$



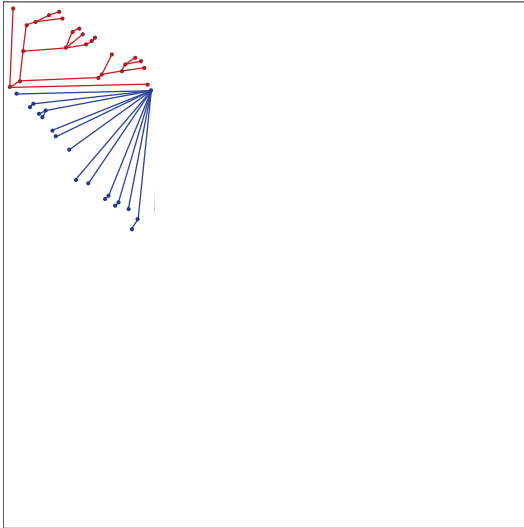
Repeatedly choose a tree and interleave its vertices with the previous one.



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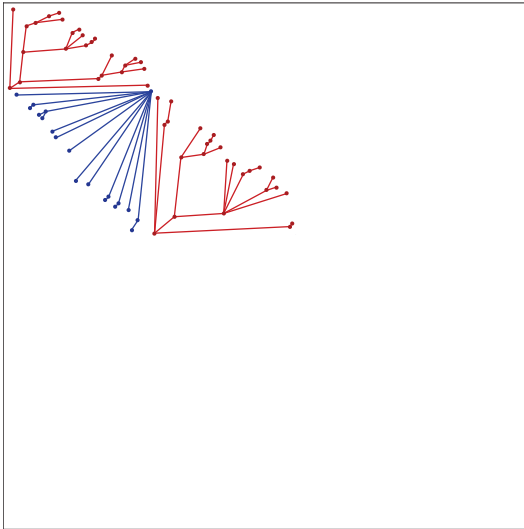
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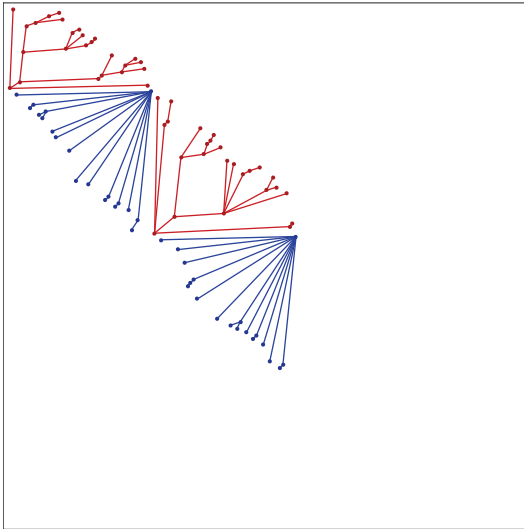
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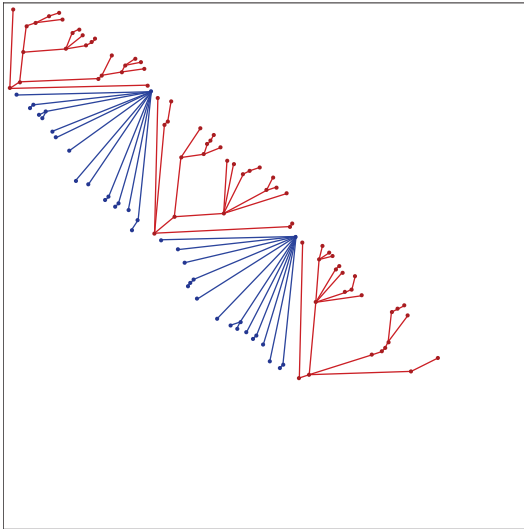
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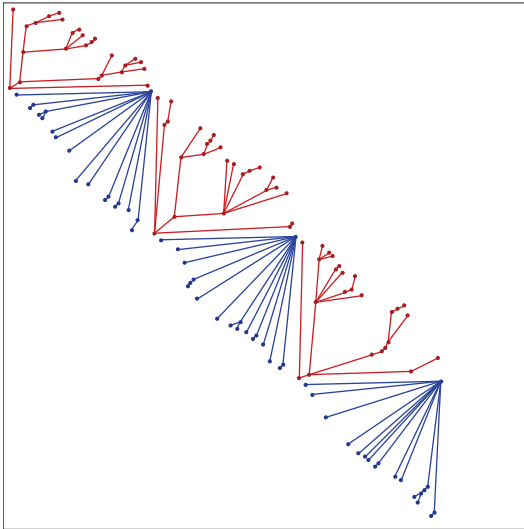
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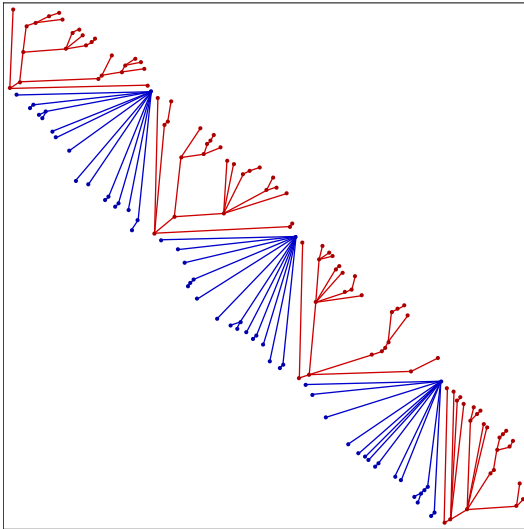
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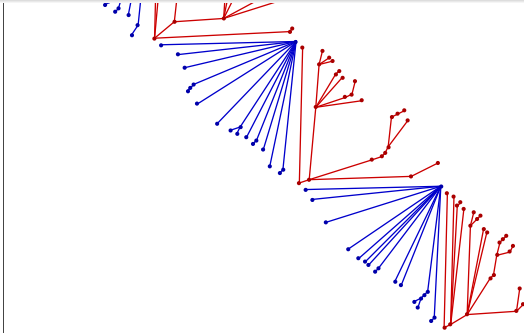
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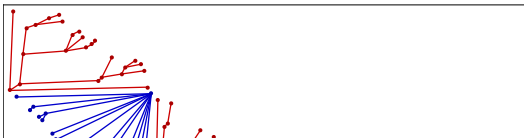
Interleaving at each step is independent of other steps.



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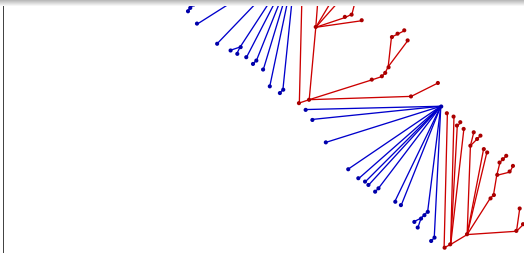
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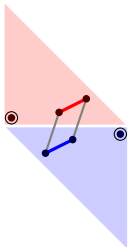
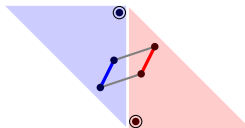
Key issue: How to avoid creation of a 1324 when interleaving.



Avoiding the creation of a 1324



Only possible causes of a 1324:

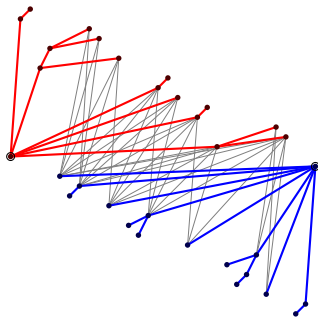


We'll focus on the horizontal case.

Avoiding the creation of a 1324: $\mathcal{W}^0$



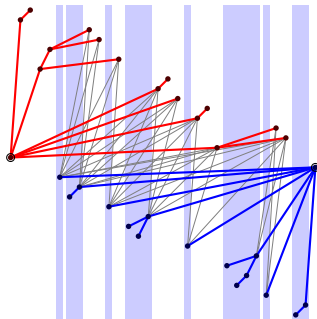
A solution: Avoid splitting principal subtrees of the blue trees.



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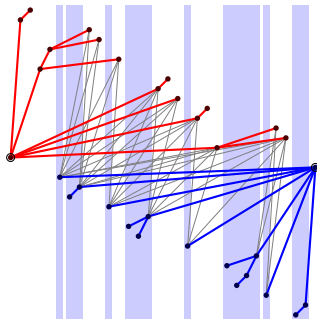
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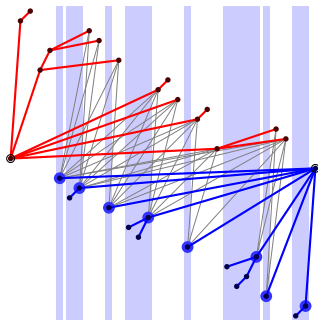
- Call principal subtree of a blue tree a **blue subtree**.

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Avoiding the creation of a 1324: $\mathcal{W}^0$



A solution: Avoid splitting principal subtrees of the blue trees.



- Call principal subtree of a blue tree a **blue subtree**.
- Call root of blue subtree a **blue root**.
- Complete freedom interleaving blue subtrees with non-root vertices of red tree.
- Denote this set $\mathcal{W}_{\lambda, \delta}^0(k)$.

Growth rate of $\mathcal{W}_{\lambda,\delta}^0(k)$



Easy to count; trees and interleavings can be chosen independently.

$$|\mathcal{W}_{\lambda,\delta}^0(k)| = R_k^{k+1} \times B_k^k \times P_k^{2k},$$

where R_k = number of distinct red trees, B_k = number of distinct blue trees, P_k = number of distinct interleavings of blue subtrees and red vertices.

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$$g_0(\lambda, \delta) = \lim_{k \rightarrow \infty} |\mathcal{W}_{\lambda,\delta}^0(k)|^{1/n(k,\lambda)} = \left(4 \frac{(2-\delta)^{(2-\delta)\lambda}}{(1-\delta)^{(1-\delta)\lambda}} \frac{(1+\delta\lambda)^{2(1+\delta\lambda)}}{(\delta\lambda)^{2\delta\lambda}} \right)^{1/(1+\lambda)}.$$

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Maximal when $\delta = \frac{2\lambda-1+\sqrt{1+4\lambda+8\lambda^2}}{2\lambda(2+\lambda)}.$

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Theorem

$\text{gr}(\text{Av}(1324)) > 9.40399.$

Let $\lambda \approx 0.61840$; $\delta \approx 0.86238$.

Interleaving beyond $\mathcal{W}^0$



Two steps:

1. Freely interleave blue roots with red vertices (as $\mathcal{W}^0$).
2. Select valid positions for non-root vertices of each blue subtree.

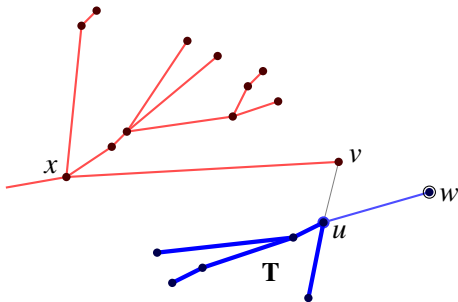
Where can the non-root vertices go?

Constraints on interleaving: red fringes



How far to the left of blue root u can the non-root vertices be placed?

- Not beyond x .

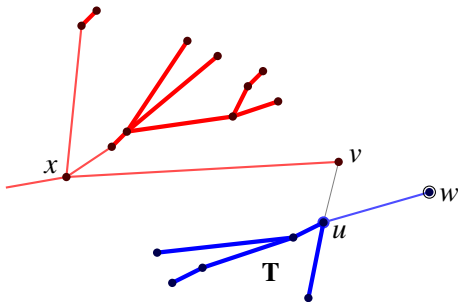


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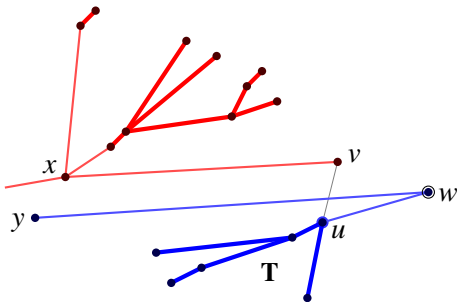
How far to the left of blue root u can the non-root vertices be placed?

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How far to the left of blue root u can the non-root vertices be placed?

- Not beyond x . Interleaved with vertices of **red forest**.
- Not beyond next blue root y .

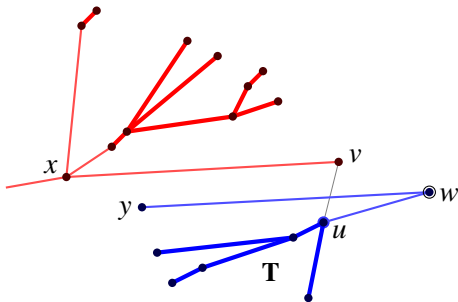


Constraints on interleaving: red fringes



How far to the left of blue root u can the non-root vertices be placed?

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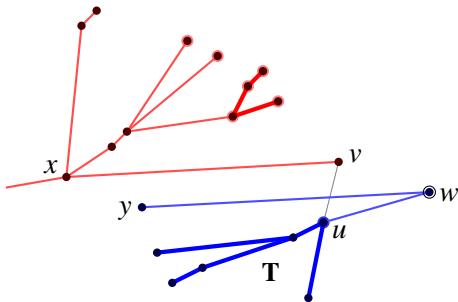


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- $\Rightarrow$ Interleaved with vertices of **red fringe** (subforest of red forest).

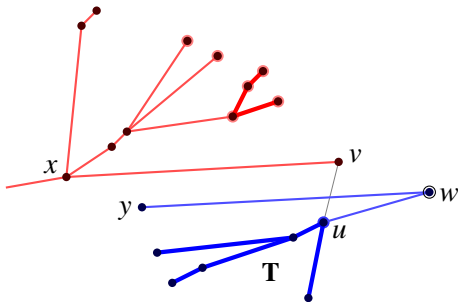


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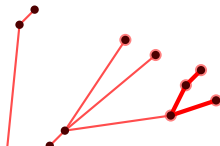
Red fringe is determined from two independent constraints.

Constraints on interleaving: red fringes

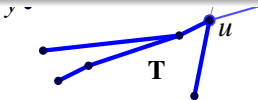


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Key fact: *Blue subtrees* are interleaved with *red fringes*.



Red fringe is determined from two independent constraints.

Interleaving blue subtrees and red fringes

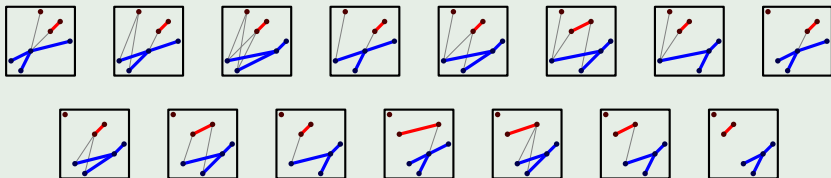


Definition ($Q(\mathbf{T}, \mathbf{F})$)

Number of distinct ways of interleaving non-root vertices of blue subtree $\mathbf{T}$ and vertices of red fringe $\mathbf{F}$ without creating a 1324.

Can calculate for small $\mathbf{T}$ and $\mathbf{F}$:

Example



$$Q(2134, 312) = 15$$

Concentration of distributions



Question

Why can we say what a “typical” large combinatorial object looks like?
Why is a single sample sufficient to illustrate the asymptotic structure?

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Why is a single sample sufficient to illustrate the asymptotic structure?

Because distributions of substructures are *concentrated*.

Example (blue subtrees)

Let $\beta_k(\mathbf{T})$ be the random variable that records the proportion of blue subtrees in a blue tree that are isomorphic to $\mathbf{T}$.

$$\lim_{k \rightarrow \infty} \mathbb{E}[\beta_k(\mathbf{T})] = \mu_{\beta}(\mathbf{T}) = \frac{(1-\delta)^{|\mathbf{T}|-1}}{(2-\delta)^{2|\mathbf{T}|-1}} \quad \text{and} \quad \lim_{k \rightarrow \infty} \mathbb{V}[\beta_k(\mathbf{T})] = 0.$$

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So, given $\varepsilon > 0$, for large enough k , the proportion of blue trees for which $|\beta_k(\mathbf{T}) - \mu_\beta(\mathbf{T})| > \varepsilon$ is less than ε .

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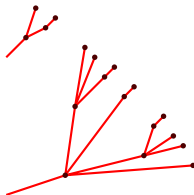
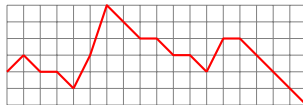
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So, given $\varepsilon > 0$, for large enough k , the proportion of blue trees for which $|\beta_k(\mathbf{T}) - \mu_\beta(\mathbf{T})| > \varepsilon$ is less than ε .

In practice, we can assume that the proportion of blue subtrees isomorphic to $\mathbf{T}$ is *exactly* $\mu_\beta(\mathbf{T})$ in *every* large blue tree.

From right to left:

- Łukasiewicz path: Steps $1, 0, -1, -2, \dots$; starts at 0; then remains +ve.



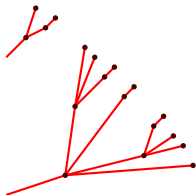
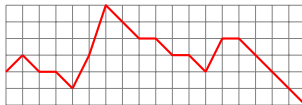
Distribution of red fringes



In bijection with patterns in Łukasiewicz paths (*pron.* “wookah-**shave**-itch”).

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Trees $\leftrightarrow$ Łukasiewicz paths:

- Height is number of components in forest induced by visited vertices.

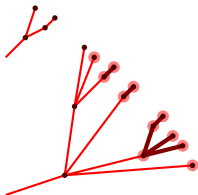
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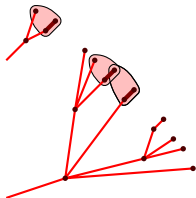
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Trees $\leftrightarrow$ Łukasiewicz paths:

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Red fringes $\leftrightarrow$ patterns in Łukasiewicz paths:

- 312 corresponds to **1,0,1** pattern.
- Multiple occurrences of a given pattern may overlap; makes it harder to enumerate.



Theorem

The number of occurrences of a fixed pattern in a Łukasiewicz path of length n exhibits a Gaussian limit distribution with mean and standard deviation asymptotically linear in n .

Patterns in Łukasiewicz paths



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Very brief outline of proof.

The bivariate generating function $L = L_\omega(z, u)$ for Łukasiewicz paths in which u marks occurrences of pattern ω satisfies

$$L = z(1 + L)^2 - (1 - u) \left(z^{m_\omega} L(1 + L)^{h_\omega} + (L - z(1 + L)^2) \hat{a}_\omega(z, 1 + L) \right).$$



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Under very general conditions on Φ , a function F satisfying $F = \Phi(z, u, F)$ is such that the parameter marked by u converges in law to a Gaussian distribution with mean and standard deviation asymptotically linear in n . □

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Corollary

The proportion of red fringes in a red tree isomorphic to $\mathbf{F}$ is concentrated.

Summing up



Asymptotically, given λ and δ , for any $\mathbf{T}$ and $\mathbf{F}$,

- Proportion of blue subtrees $\cong \mathbf{T}$ has concentrated distribution.
- Proportion of red fringes $\cong \mathbf{F}$ has concentrated distribution.
- Red and blue trees can be chosen independently.
- So, the proportion of blue subtrees $\cong \mathbf{T}$ with red fringe $\cong \mathbf{F}$ also has concentrated distribution; let $\mu(\mathbf{T}, \mathbf{F})$ be the limiting mean.

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- $Q(\mathbf{T}, \mathbf{F})$: number of valid ways of interleaving $\mathbf{T}$ and $\mathbf{F}$.
- $g_0(\lambda, \delta)$: growth rate from interleaving blue roots.

Proposition

If $\mathcal{S}$ is a finite set of pairs $(\mathbf{T}, \mathbf{F})$, then

$$g(\lambda, \delta) \geq g_0(\lambda, \delta) \times \prod_{(\mathbf{T}, \mathbf{F}) \in \mathcal{S}} Q(\mathbf{T}, \mathbf{F})^{2\delta\lambda\mu(\mathbf{T}, \mathbf{F})/(1+\lambda)}.$$

Summing up



Theorem

$\text{gr}(\text{Av}(1324)) > 9.81.$

Proof.

For each $N > 0$, the growth rate is at least

$$g_N(\lambda, \delta) = g_0(\lambda, \delta) \times \prod_{|\mathbf{T}|+|\mathbf{F}| \leq N} Q(\mathbf{T}, \mathbf{F})^{2\delta\lambda\mu(\mathbf{T}, \mathbf{F})/(1+\lambda)}.$$

Using Mathematica to calculate $Q(\mathbf{T}, \mathbf{F})$ and $\mu(\mathbf{T}, \mathbf{F})$ for 1.6 million pairs, and then to apply numerical maximisation over values of λ and δ yields

$$g_{14}(\lambda, \delta) > 9.81056$$

with $\lambda \approx 0.69706$ and $\delta \approx 0.75887$. □

The value of $\lim_{N \rightarrow \infty} \max_{\lambda, \delta} g_N(\lambda, \delta)$ is probably not far from 9.82.

Thanks for listening!

