Unbalanced Wilf-equivalences

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Joint work with Jonathan Bloom, Vít Jelínek and Jay Pantone

Pattern

Definition

A pattern is an equivalence class of sequences under order isomorphism.

Example

246135 contains pattern 3412 since it contains subsequence 4613, which is order isomorphic to 3412 (is an instance of pattern 3412). On the other hand, 246135 has no instances of pattern 2143, so 246135 avoids 2143.

Notation

$\mathrm{Av}_n(\tau) = \{\pi \in S_n \mid \pi \text{ avoids } \tau\}$ $\quad$ $\mathrm{Av}_n(T) = \bigcap_{\tau \in T} \mathrm{Av}_n(\tau)$

$\mathrm{Av}(T) = \bigcup_{n \in \mathbb{N}} \mathrm{Av}_n(T)$

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(Sets of) Patterns τ_1 and τ_2 are called Wilf-equivalent, denoted $\tau_1 \sim \tau_2$, if $|\mathrm{Av}_n(\tau_1)| = |\mathrm{Av}_n(\tau_2)|$ for all $n \in \mathbb{N}$. A Wilf class of τ is the collection of all (sets of) patterns Wilf-equivalent to τ.

Example

All permutations in S_3 are Wilf-equivalent (Knuth, 1968), and $|Av_n(\tau)| = C_n$, the nth Catalan number, for any $\tau \in S_3$.

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Wilf classes in S_4

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In S_4, there are 7 symmetry classes and 3 Wilf classes (Gessel, 1990; West, 1990; Stankova, 1994; Bóna, 1997).

τ	$(\lvert Av_n(\tau)\rvert)_{n\geq 0}$
1423 2413	1, 1, 2, 6, 23, 103, 512, 2740, 15485, . . .
1234 1243 1432 2143	1, 1, 2, 6, 23, 103, 513, 2761, 15767, . . .
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LR-min and RL-max

For a permutation $\pi \in S_n$, find its LR-minima and RL-maxima, then find the cells in its permutation diagram that are above and to the right of an LR-minimum and below and to the left of an RL-maximum. This yields a skew-Ferrers board B_π.

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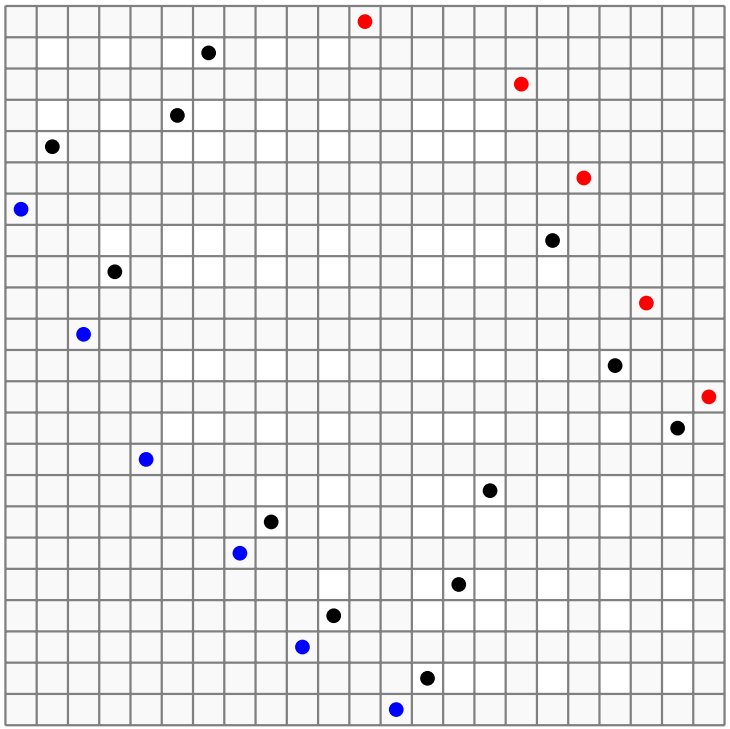

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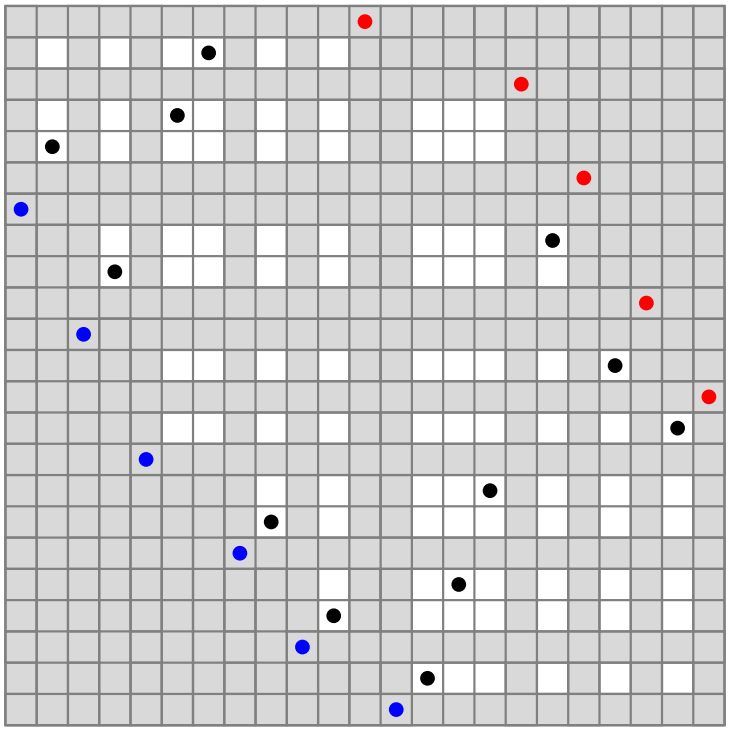

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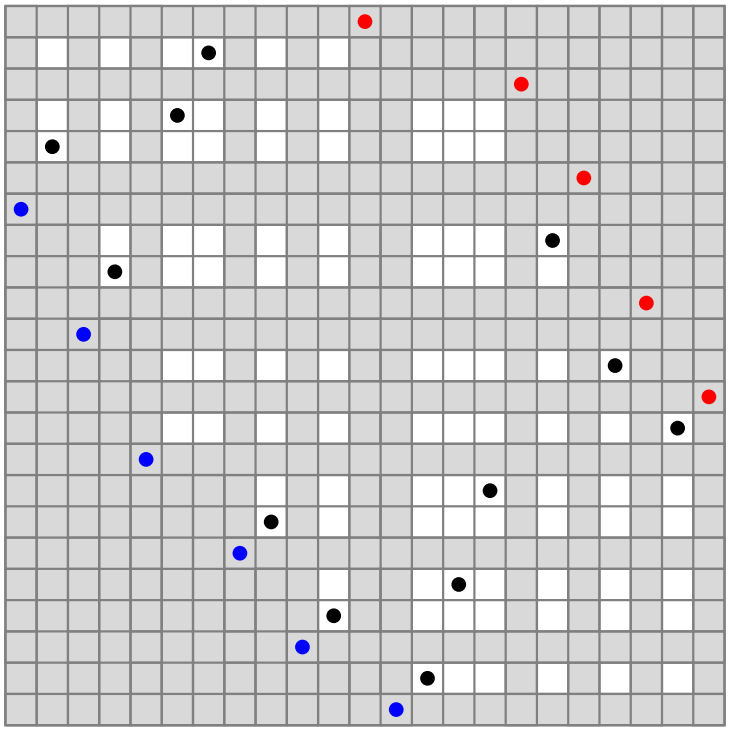

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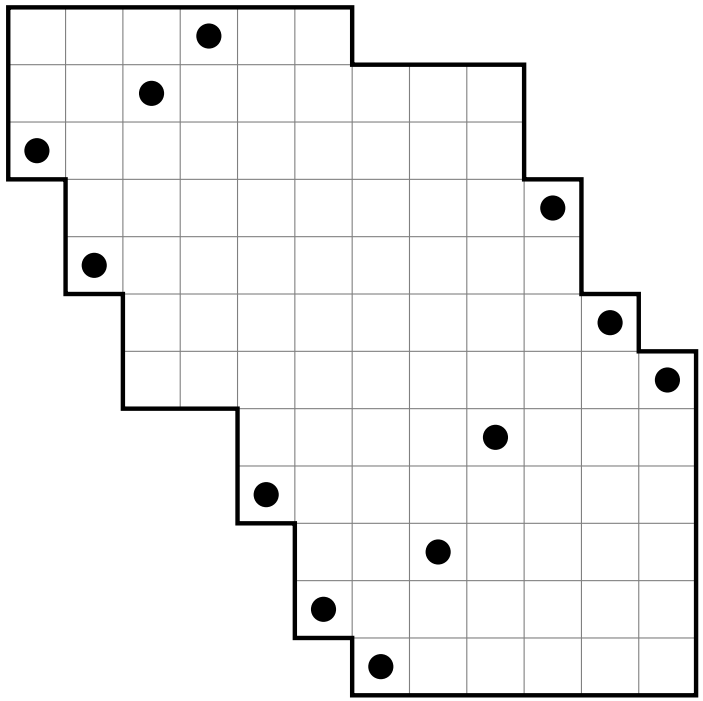

B_π

Skew-boards

B_σ has $\#rows = \#columns$ and contains the antidiagonal. Call such a skew-shape permissible. A filling of a permissible board B is a permutation diagram with all entries inside B.

Example

B_{1234}

B_{1324}

Lemma (Bona, 1997)

Each permissible board B has a unique filling that avoids B_{1234} (along the antidiagonal) and at least one filling that avoids B_{1324}.

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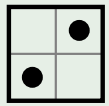

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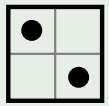

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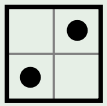

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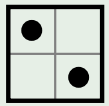

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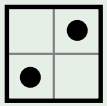

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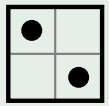

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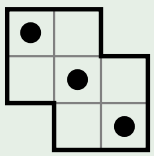

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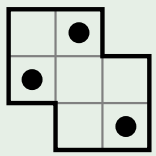

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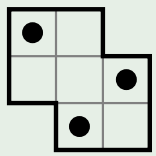

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Some notation and facts

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$I_2 = B_{1234}$, $J_2 = B_{1324}$, $\rho = B_{3416725}$.

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Each permissible board B has a unique I_2-avoiding filling, the one along the antidiagonal of B.

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Unbalanced Wilf-equivalence

Theorem (B., Pantone, 2014; Jelínek, 2008)

$1234 \sim (1324, 3416725)$

Definition

We call a Wilf-equivalence unbalanced if the two sets of forbidden patterns do not contain the same number of patterns of each length.

The type of an unbalanced Wilf-equivalence is defined by the lengths of the patterns involved. For example, the above Wilf-equivalence is of the type $(4) \sim (4, 7)$.

1 prior result: Atkinson, Murphy, Ruškuc, 2002: a Wilf-equivalence of the type $(4) \sim (4, 6, 8, \dots)$

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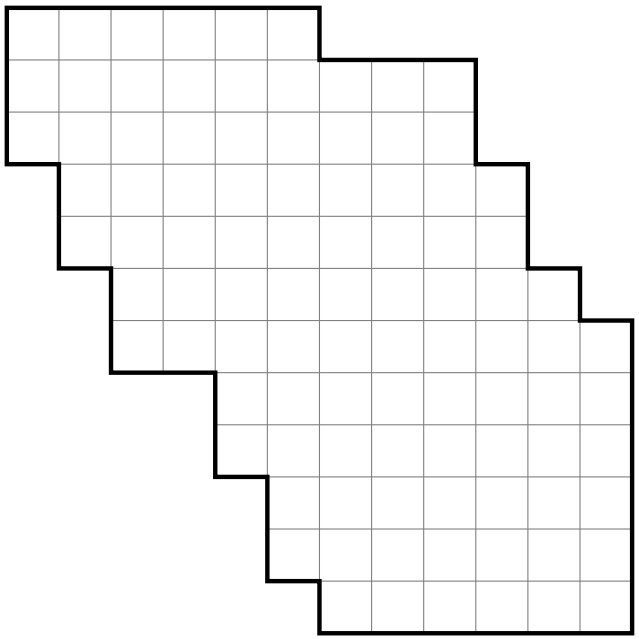

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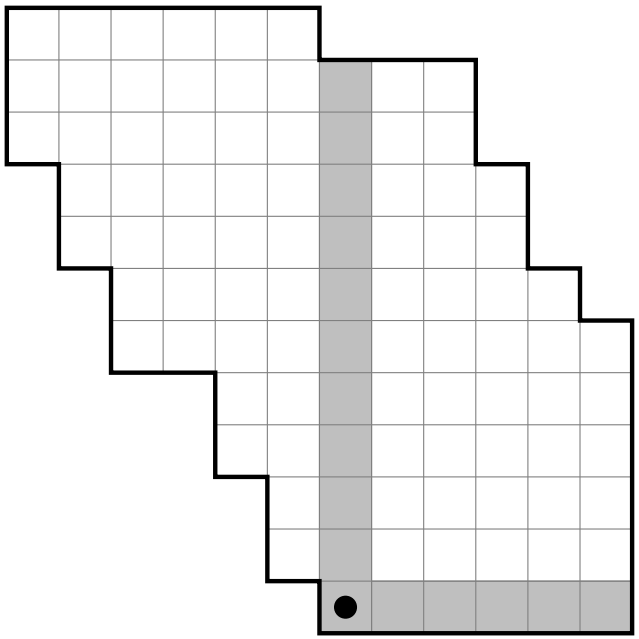

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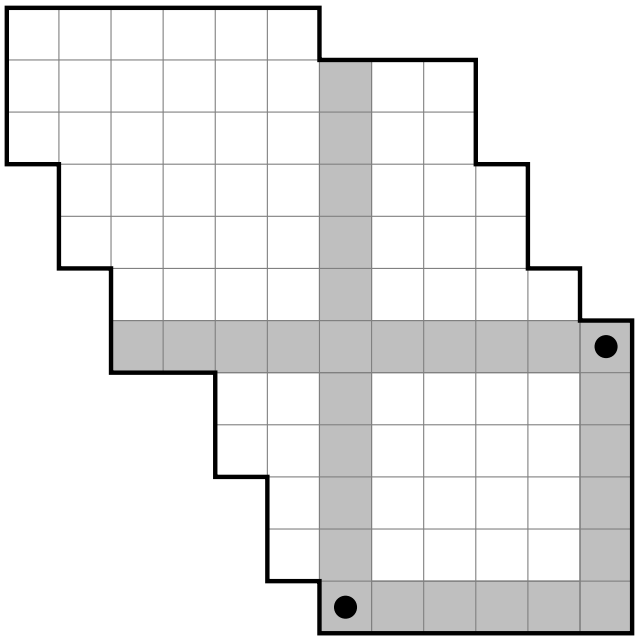

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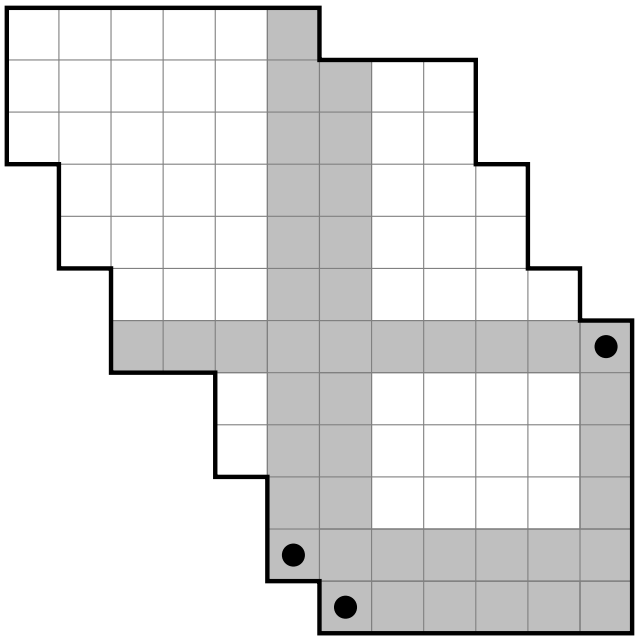

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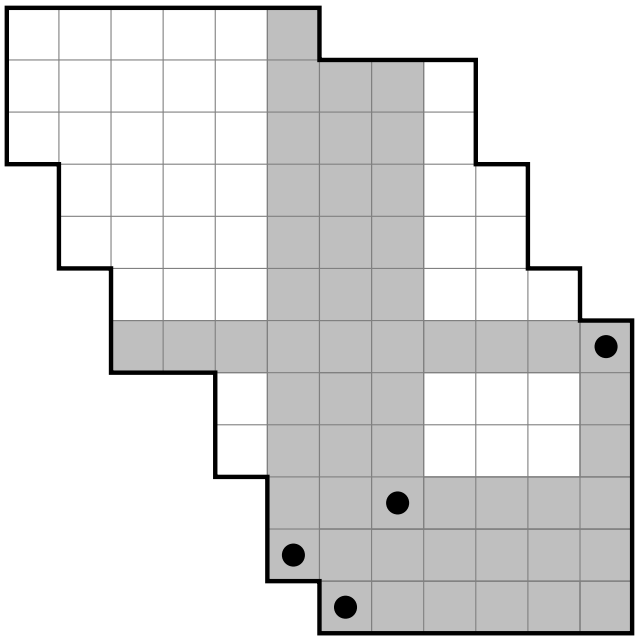

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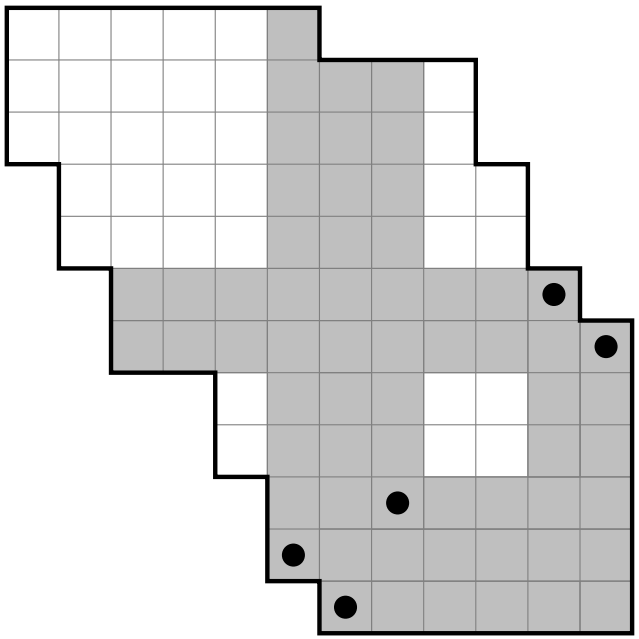

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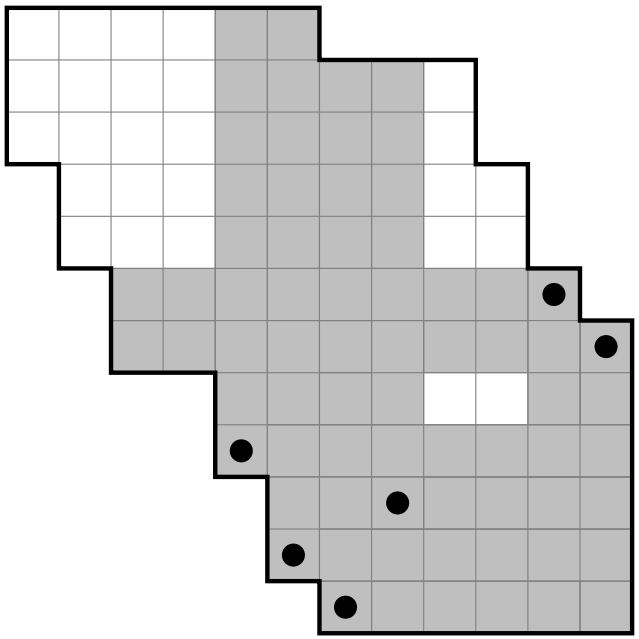

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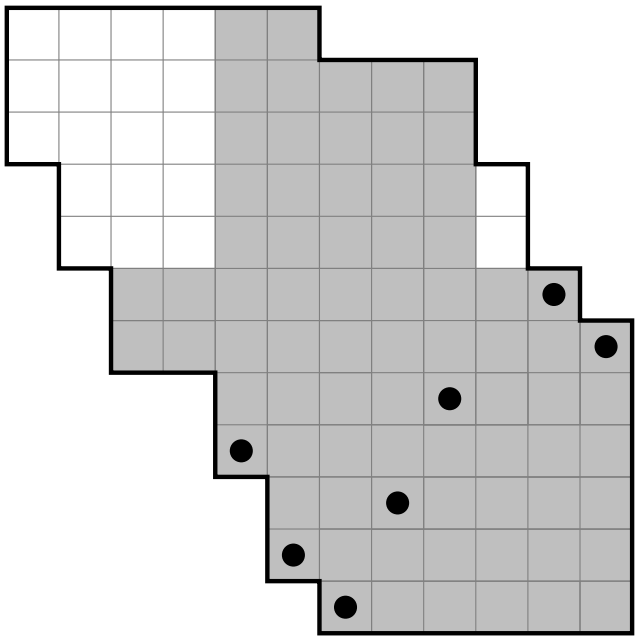

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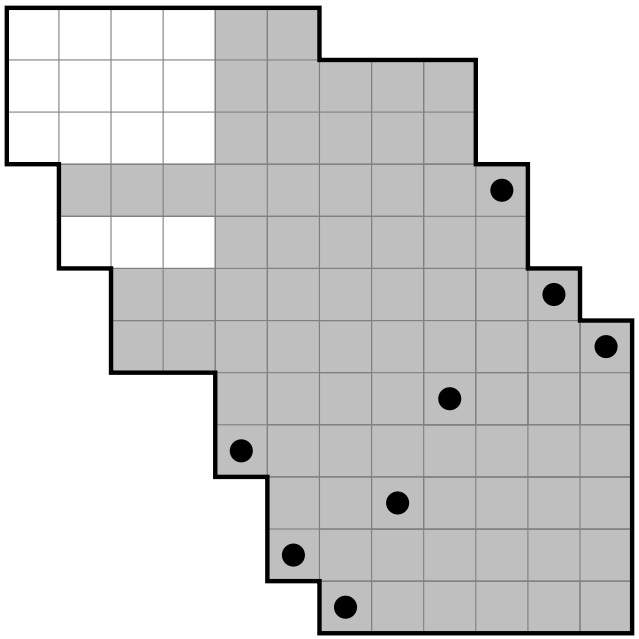

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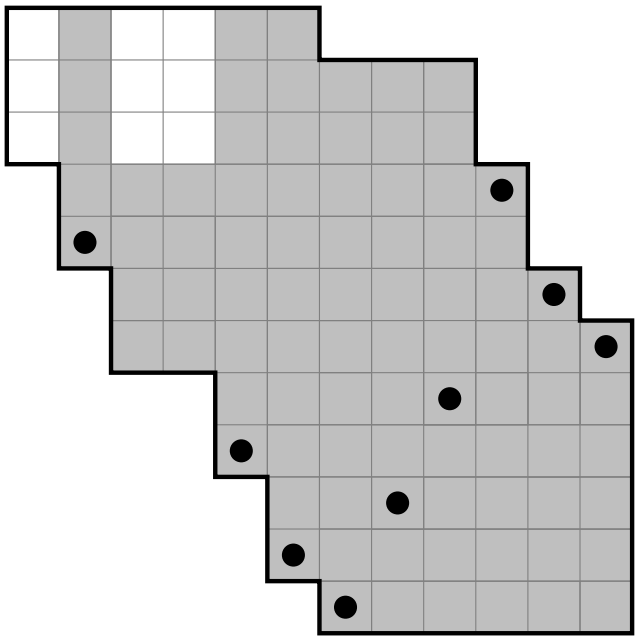

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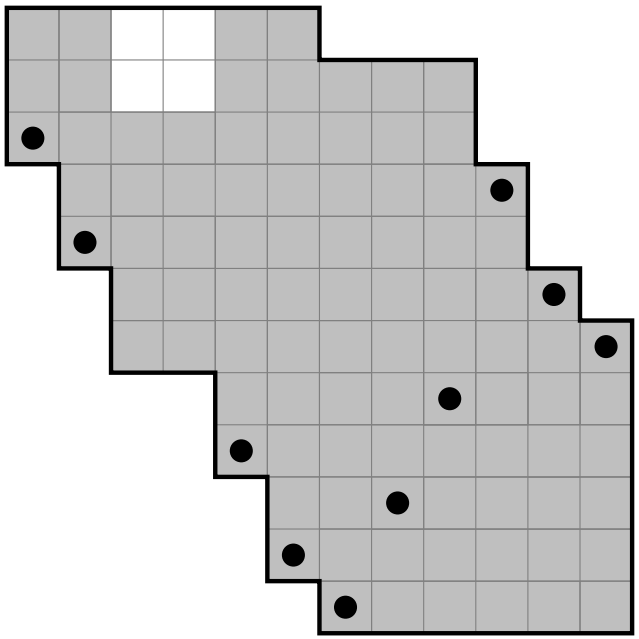

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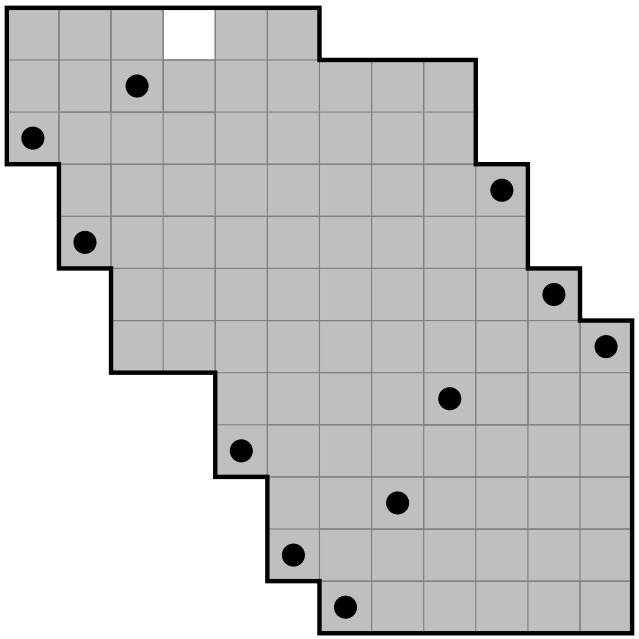

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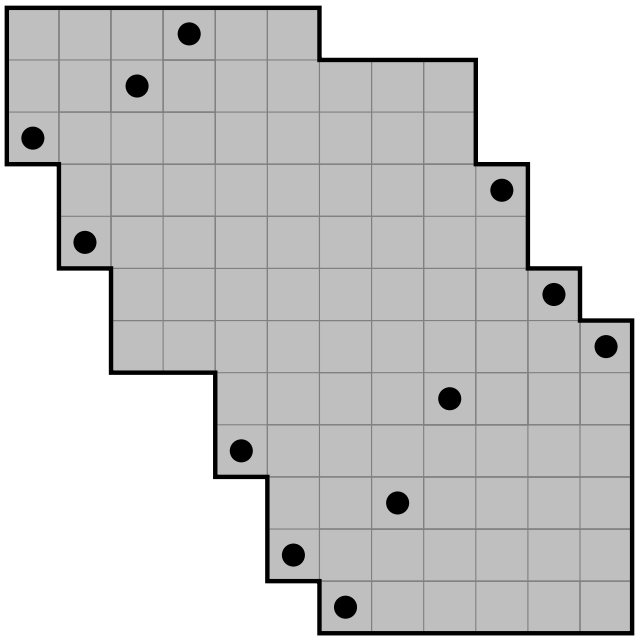
Each permissible board B *has a unique* (J_2, ρ) *-avoiding filling. It is the lexicographically smallest* J_2*-avoiding filling of* B.



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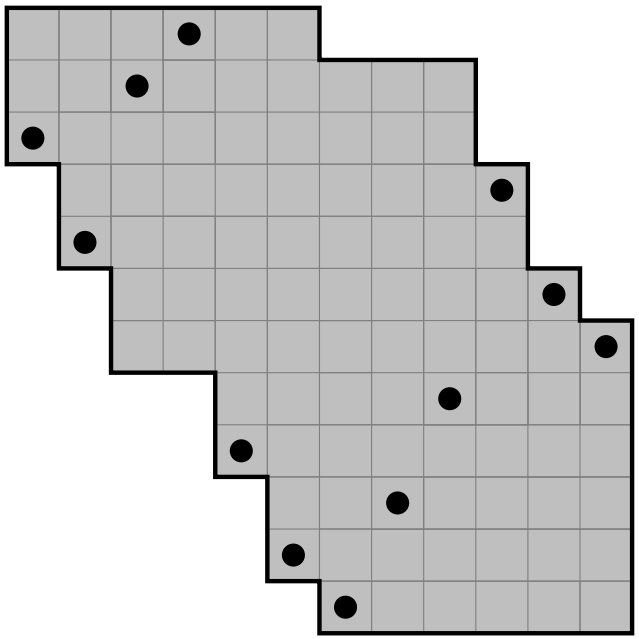

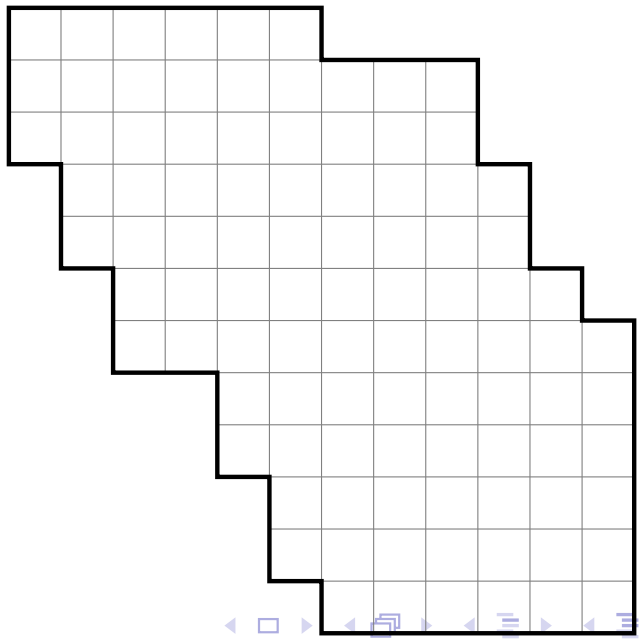

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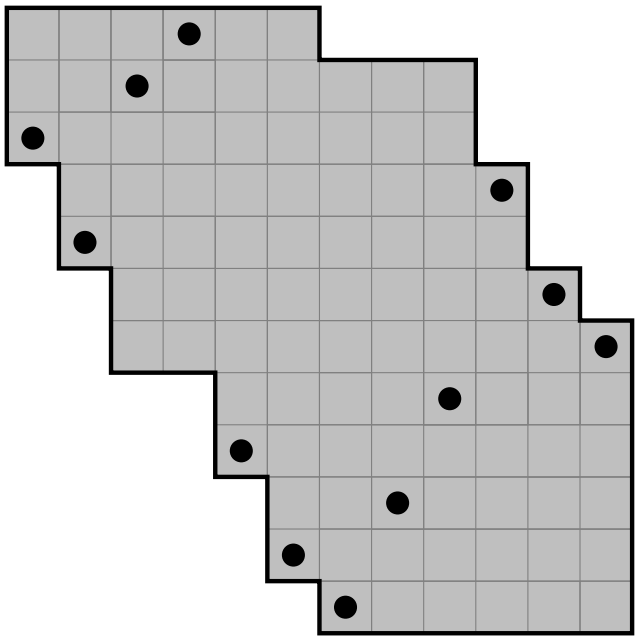

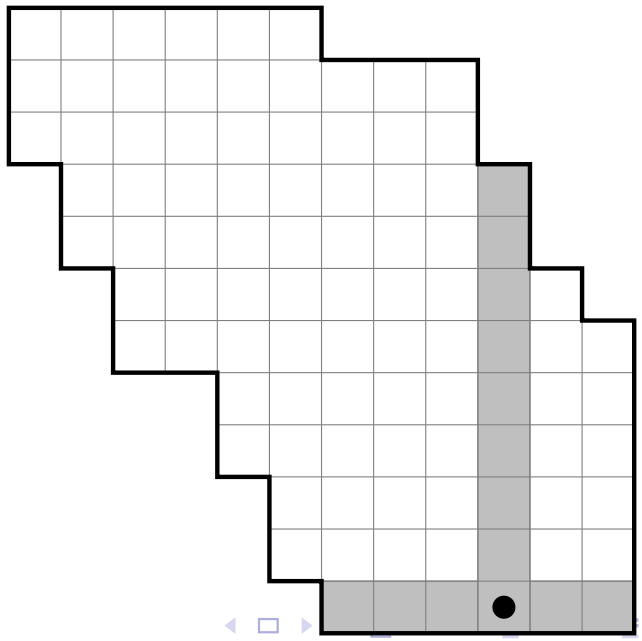

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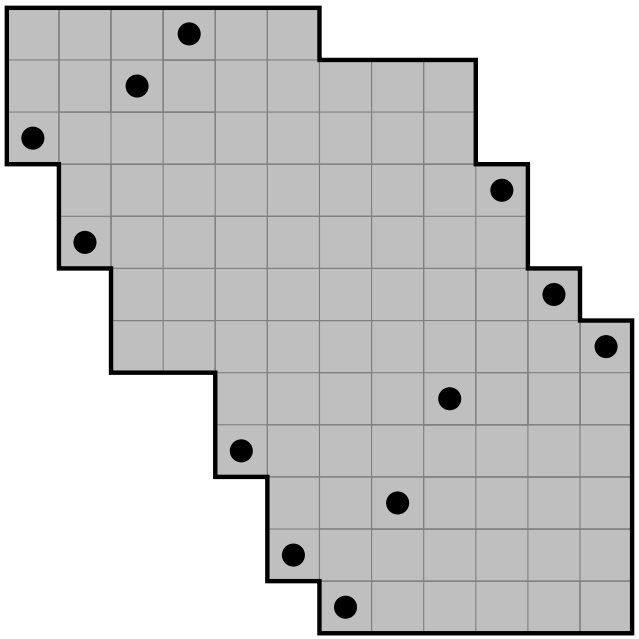

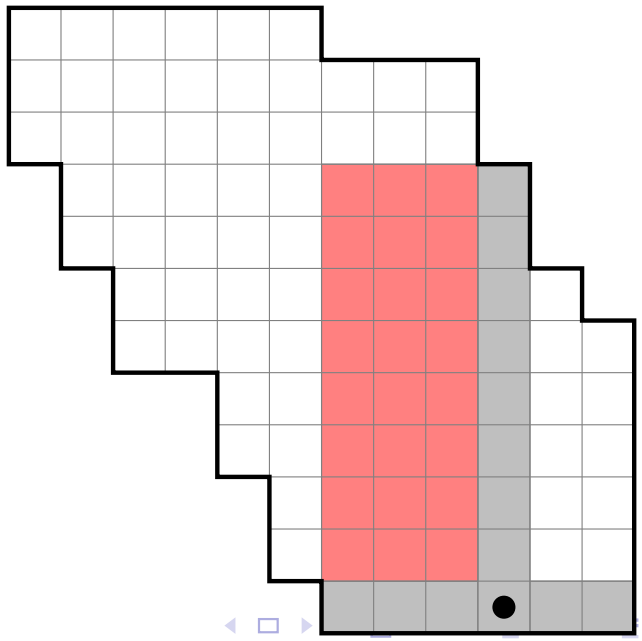

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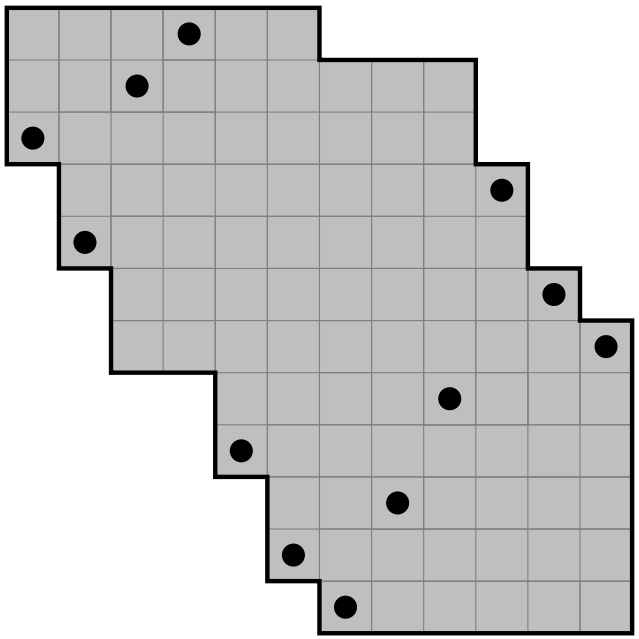

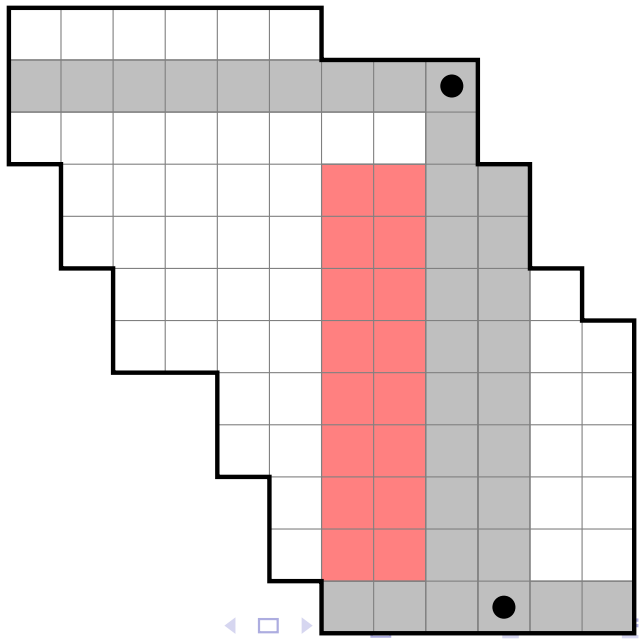

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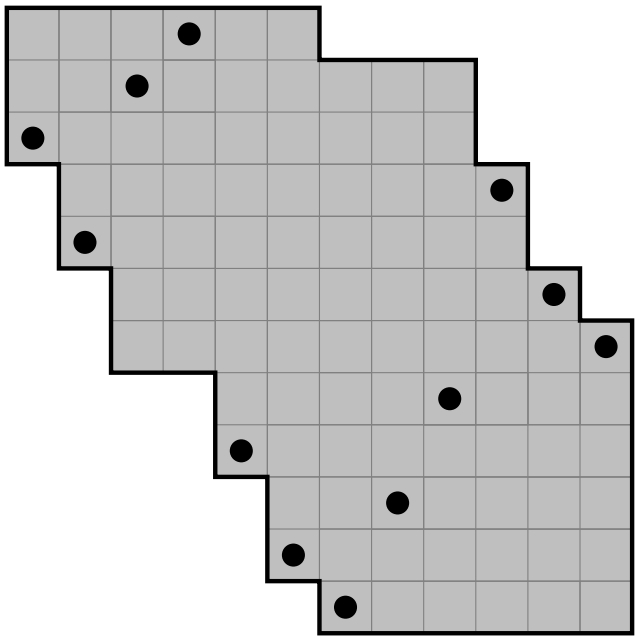

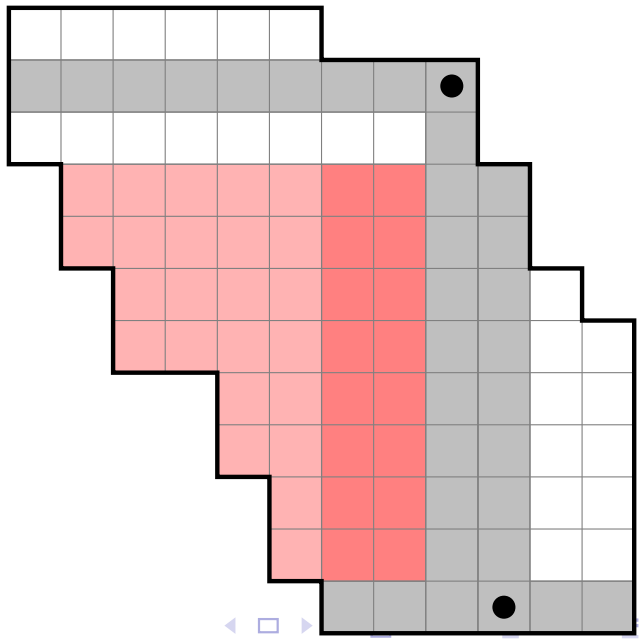

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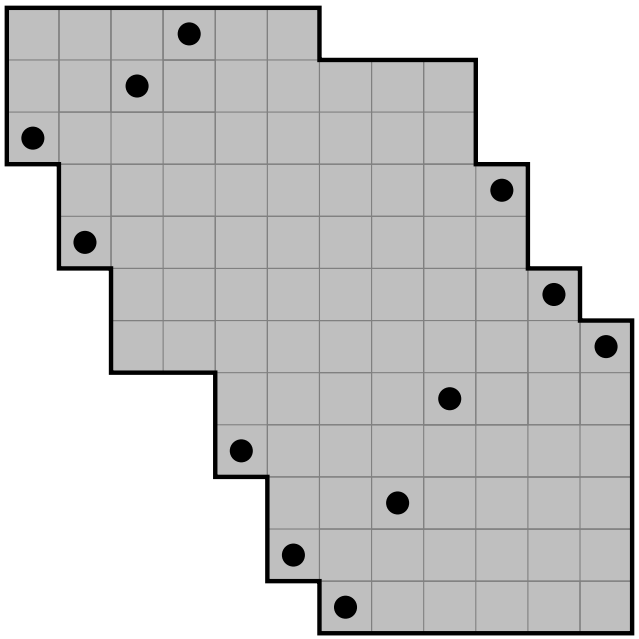

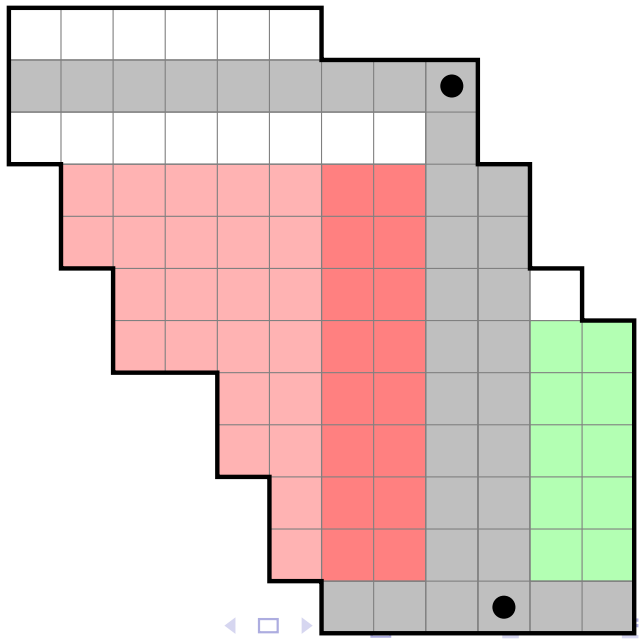

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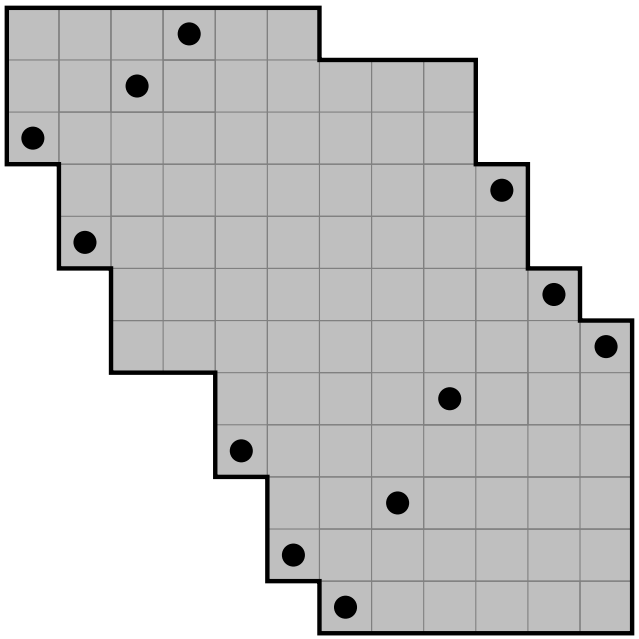

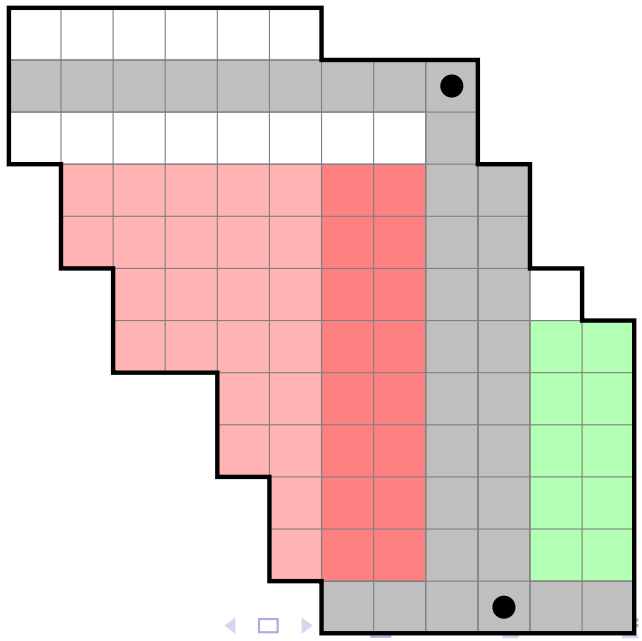

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An involution is a permutation π such that $\pi^{-1} = \pi$. Let $\mathcal{I}_n(\tau)$ be the set of involutions avoiding pattern (or set of patterns) τ.

Theorem (B., Pantone, 2014)

$4321 \sim (4231, 5276143)$ on involutions, and
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One more time

τ	$(\lvert Av_n(\tau)\rvert)_{n\geq 0}$
1423 2413	1, 1, 2, 6, 23, 103, 512, 2740, 15485, ...
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Q: Where does the extra 1 come from?

Conjecture (B., Pantone, 2014)

$2413 \sim (2143, 246135)$

Checked for permutations of length ≤ 17.

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There are no other $(4) \sim (4, k)$ *Wilf-equivalences for any* k.

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Conjecture (Egge, 2011)

$|Av_n(2143, 3142, \tau)| = r_{n-1}$, *the* $(n-1)$*-st large Schröder number, for each* τ *among*

$$246135,\ 254613,\ 263514,\ 524361,\ 546132.$$

These give rise to many $(4, 4) \sim (4, 4, 6)$ Wilf-equivalences, for example,

$$(2413, 3142) \sim (2143, 3142, 246135).$$

All cases are now proven:

- (B., Pantone, 2014): 246135.
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Intervals and simple permutations

Definition

A permutation π contains an interval if $\pi([i,j]) = [k,l]$ for some i,j,k,l with $j-i = l-k$.

Trivially, every $\pi \neq \emptyset$ contains intervals of length 1 and $|\pi|$.

Definition

A permutation π is simple if π has no nontrivial intervals.

Example

1, 12, 21, 2413, 3142, 24153, 31524, 35142, 42513, . . .

Notation

$\mathrm{Si}(S) =$ simple permutations in set S.

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$3142[123, 1, 21, 312] = 567\ 1\ 98\ 423$

Definition

- Sum-decomposable permutation $=$ inflation of 12
- Skew-decomposable permutation $=$ inflation of 21

Theorem (Albert and Atkinson, 2005)

Given a permutation π, there exists a unique simple permutation σ such that $\pi = \sigma[\tau_1, \ldots, \tau_k]$. When $\sigma \notin \{12, 21\}$, the intervals $\tau_1, \ldots, \tau_k$ are uniquely determined. When $\sigma = 12$ (resp., $\sigma = 21$), the intervals are unique if we require the first of the two intervals to be sum (resp., skew) indecomposable.

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Given a permutation π, there exists a unique simple permutation σ such that $\pi = \sigma[\tau_1, \ldots, \tau_k]$. When $\sigma \notin \{12, 21\}$, the intervals $\tau_1, \ldots, \tau_k$ are uniquely determined. When $\sigma = 12$ (resp., $\sigma = 21$), the intervals are unique if we require the first of the two intervals to be sum (resp., skew) indecomposable.

Simple permutations

Lemma (B., Pantone, 2014)

$$\mathrm{Si}(\mathrm{Av}(2143, 3142, 246135)) = \mathrm{Si}(\mathrm{Av}(2143, 3142, 3412))$$

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Proof.

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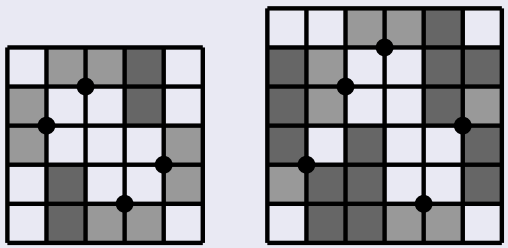

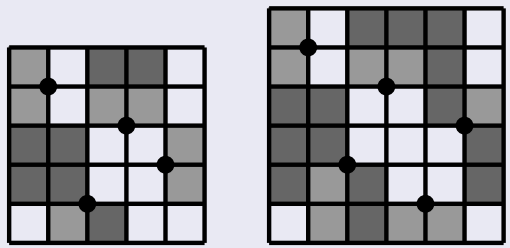

From simples to all

Every permutation falls into exactly one of the following categories:

- the single permutation of length 1,
- sum decomposable permutations (inflations of 12),
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Albert, Vatter, 2013: $\mathrm{Av}(2143, 3412)$ are the skew-merged permutations: exactly half of them contains 3142 and avoids 2413, the other half contains 2413 and avoids 3142.

Entries in the increasing subsequence can only be inflated by increasing permutations (a decrease would form an instance of 2143). Entries in the decreasing subsequence and the (optional) central point can all be inflated by any permutation in the class.

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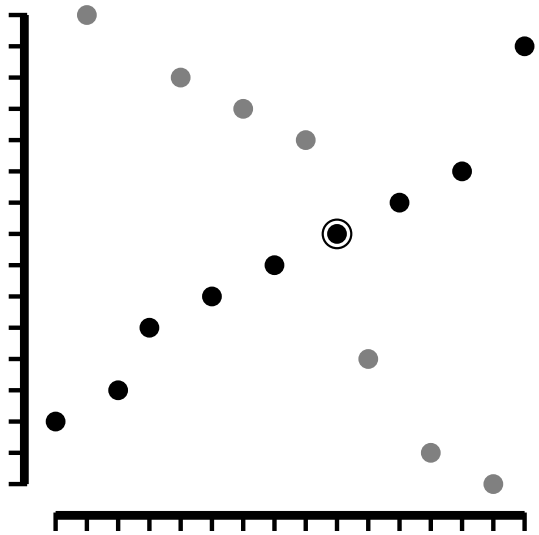

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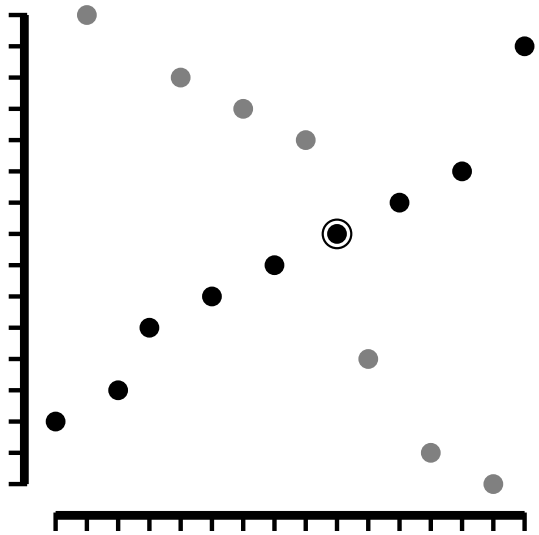

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where

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Some further directions

- Prove that $2413 \sim (2143, 246135)$.
- Generalize $(4) \sim (4, 7)$ and $(4) \sim (4, 6)$ to $(r) \sim (r, \dots)$ for arbitrary $r \geq 4$.
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Generalization of $(4) \sim (4, 7)$

Red and blue entries, generalized

Let σ and τ be any nonempty permutations.

For example, in the proof of $1234 \sim \{1324, 3416725\}$, we had $\sigma = 1$ and $\tau = 1$.

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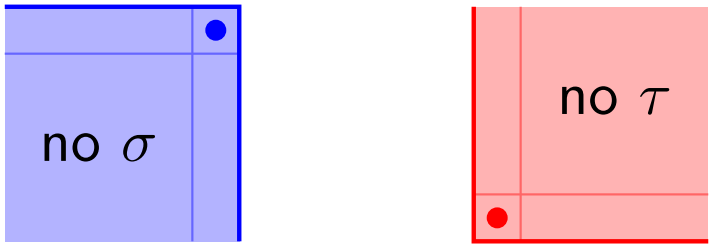

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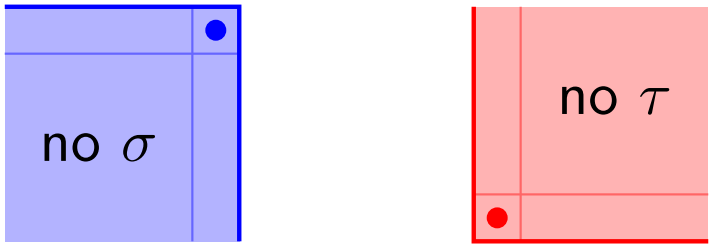

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Wrapping and unwrapping

Definition

The (σ, τ)-unwrapping of a permutation π is a (permissible) board $B_\pi^{\sigma,\tau}$ obtained from the permutation diagram of π by deleting red and blue entries along with their rows and columns and the union of all red and blue regions.

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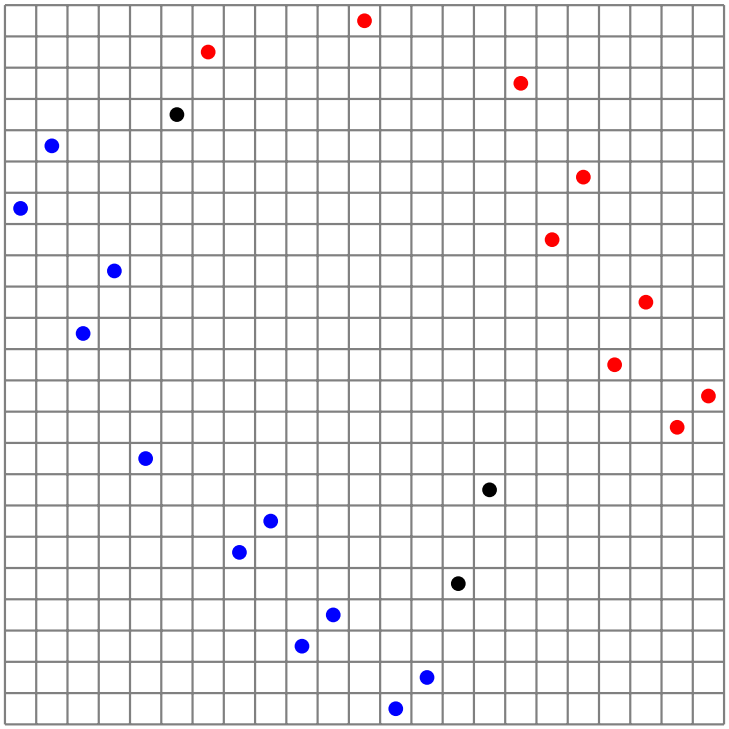

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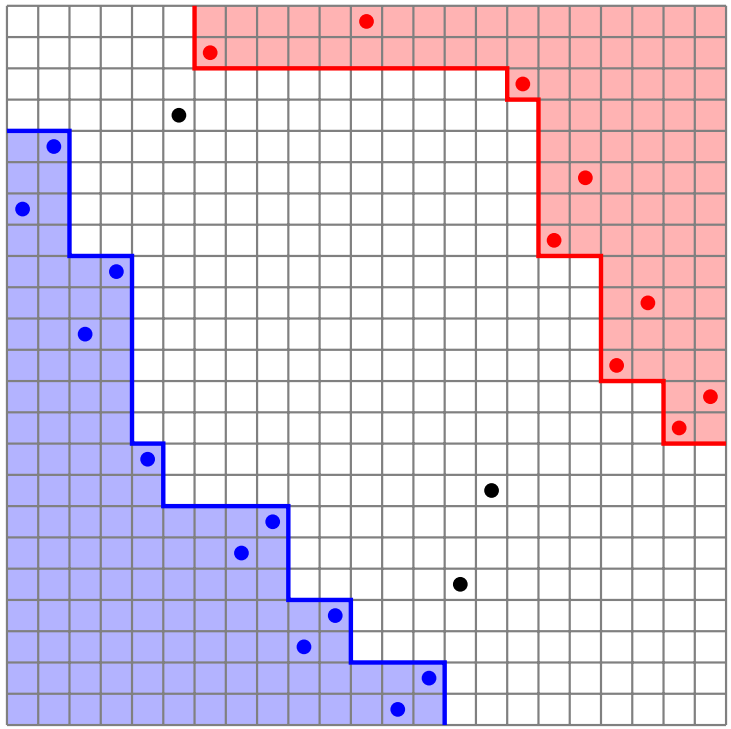

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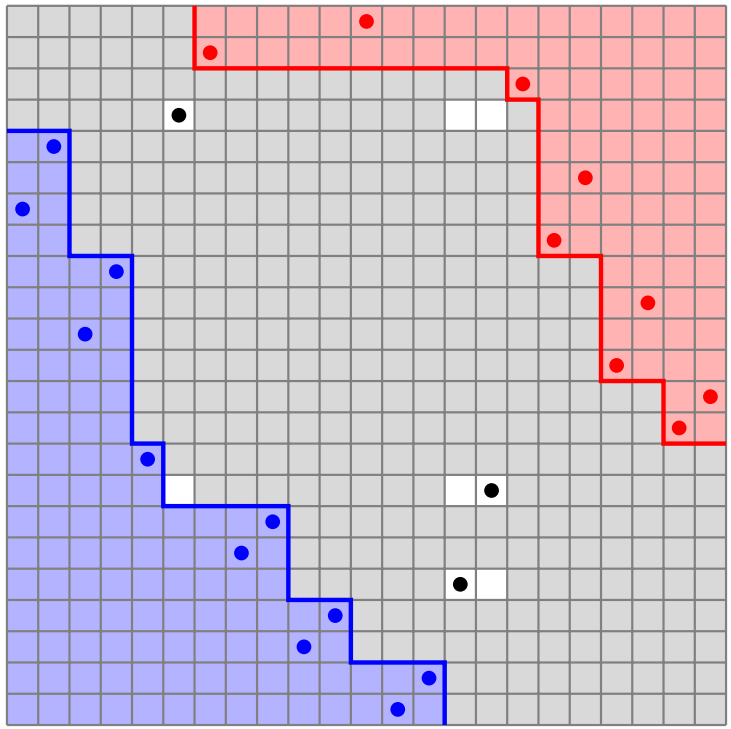

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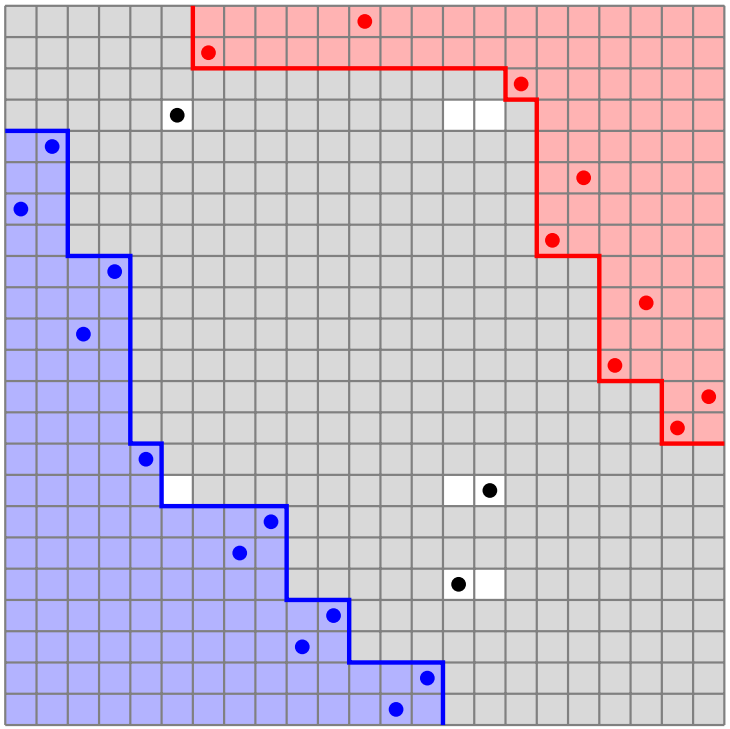

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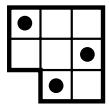

$B_\pi^{\sigma,\tau}$

Generalization of $(4) \sim (4,7)$

Define $\pi_1 \oplus \pi_2 = 12[\pi_1, \pi_2]$, $\pi_1 \ominus \pi_2 = 21[\pi_1, \pi_2]$.

Theorem (Jelínek, 2014)

For any nonempty permutations σ, τ,

$$\sigma \oplus I_2 \oplus \tau \sim \{\sigma \oplus J_2 \oplus \tau, \mathrm{mw}^{\sigma,\tau}(\rho)\},$$

where $\mathrm{mw}^{\sigma,\tau}(\rho)$ is the set of minimal (σ,τ)-wrappings of ρ.

Note that any $\pi \in \mathrm{mw}^{\sigma,\tau}(\rho)$ has

$$|\sigma| + |\tau| + 5 \leq |\pi| \leq 2|\sigma| + 2|\tau| + 3.$$

Two examples

Example

$\sigma = 1$, $\tau = 12$:

$$12345 \sim \{13245, 34167258, 341689257, 341782956, 341682957, 341789256\}.$$

$\sigma = 1$, $\tau = 21$:

$$12354 \sim \{13254, 34168275, 341687295, 341792658, 341798265\}.$$

Note that $12345 \sim 12354$, so this yields a $(5, 8, 9, 9, 9, 9) \sim (5) \sim (5, 8, 9, 9, 9)$ unbalanced Wilf-equivalence.
Note also that $13245 \sim 13254$.

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