

Inversions on Permutations Avoiding Consecutive Patterns

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The Basics

A permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ of length n will simply be any way to write the numbers 1 through n in some order. We use S_n to denote the group of all permutations of length n and $|S_n| = n!$.

Definition

Given a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$, we define an **inversion** to be a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$.

Definition

The **inversion statistic**, inv , is given by

$\text{inv}(\pi) =$ the total number of inversions in π

Example

$\pi = 328574619$, $\text{inv}(\pi) = 15$

Classical Pattern Avoidance

Definition

Let $\pi \in S_n$. We say “ π **contains** σ **as a pattern**” if π has a subsequence that is order isomorphic to σ .

Definition

We say that π **avoids** σ **as a pattern** if π contains no subsequence order isomorphic to σ .

Classical vs. Generalized Pattern Avoidance

- **Classical:** σ is written as a permutation of the numbers $1, 2, \dots, k$ and elements of the pattern need not appear as adjacent in π .
- **Generalized Pattern Avoidance:** σ is written as a list of the numbers $1, 2, \dots, k$ **with dashes inserted between elements that need not appear as adjacent in π and no dashes otherwise.**

Example

$\pi = 45213$ contains $3 - 12$ and $3 - 1 - 2$ but avoids 312 .

- A classical pattern is a generalized pattern with all internal dashes.
- A generalized pattern with no internal dashes is called a **consecutive** pattern.

Example

Find all permutations in S_4 that avoid 312 as a consecutive pattern.

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Find all permutations in S_4 that avoid 312 as a consecutive pattern.

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

Permutations in bold contain 312.

Wilf-equivalence

Definition

Let Π be a collection of generalized patterns. We let $\text{Av}_n(\Pi)$ denote the set of permutations in S_n that avoid every pattern in Π .

Example

$$|\text{Av}_4(312)| = 16.$$

Definition

We say that two sets of generalized permutation patterns Π and Π' are *Wilf equivalent* if $|\text{Av}_n(\Pi)| = |\text{Av}_n(\Pi')|$.

st-Wilf Equivalence

Sagan and Savage [14] recently defined a q -analogue of Wilf equivalence by considering any permutation statistic st from $\uplus_{n \geq 0} S_n \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, and letting

$$F_n^{\text{st}}(\Pi; q) = \sum_{\sigma \in \text{Av}_n(\Pi)} q^{\text{st}(\sigma)}.$$

For Π and Π' subsets of permutations, they defined Π and Π' to be *st-Wilf equivalent* if $F_n^{\text{st}}(\Pi; q) = F_n^{\text{st}}(\Pi'; q)$ for all $n \geq 0$.

We use the same definition for Π and Π' sets of generalized permutation patterns.

- Dokos et al.[7] give a thorough investigation of st-Wilf equivalence for classical patterns of length 3 for both the major index, maj, and the inversion statistic, inv.
- Elizalde-Noy, Kitaev, Kitaev-Mansour, Aldred-Atkinson-McCaughan [1, 9, 10, 12, 13] collectively accomplished a comprehensive enumeration of permutations avoiding multiple consecutive patterns of length three.

The main focus of the present work is the inversion statistic on permutations avoiding sets of consecutive patterns of length three.

Inversion Polynomial $I_n(\Pi; q)$

Definition

Let Π be a set of (generalized) patterns. The **inversion polynomial** on $\text{Av}_n(\Pi)$ is given by

$$I_n(\Pi; q) = \sum_{\sigma \in \text{Av}_n(\Pi)} q^{\text{inv}(\sigma)}$$

Note that $I_n(\Pi; 1) = |\text{Av}_n(\Pi)|$.

Example

$$\begin{aligned} I_4(\{3-1-2\}; q) &= 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6 \\ I_4(\{312\}; q) &= 1 + 3q + 3q^2 + 4q^3 + 3q^4 + q^5 + q^6 \end{aligned}$$

Fibonacci Tableaux

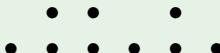
Recall that the number of ways to write a positive integer n as a sum of 1's and 2's is a Fibonacci number.

Definition

A **Fibonacci shape** of size n is an ordered list of 1's and 2's which sums to n . The **Ferrers diagram** for a Fibonacci shape is formed by replacing each 1 with a single dot and each 2 with two dots.

Example

The Fibonacci shape 122121 has size 9 and Ferrers diagram:



Fibonacci Tableaux

Definition

A **standard Fibonacci tableau** for a Fibonacci shape μ of size n is a filling of the Ferrers diagram of μ with the numbers $1, 2, \dots, n$ so that the bottom row decreases from left to right and each column decreases from bottom to top.

Example

A standard Fibonacci tableau of shape $\mu = 122121$ is

	3	4		2	
9	8	7	6	5	1

Fibonacci Tableaux

	3	4		2	
9	8	7	6	5	1

Definition

The **column-reading word** $w_c(T)$ of a standard Fibonacci tableau T is obtained by reading the columns of T from right to left, bottom to top.

Example

For the tableau above, $w_c(T) = 152674839$.

Definition

The **inversion number** of a standard Fibonacci tableau T is defined as $\text{inv}(T) := \text{inv}(w_c(T))$.

Fibonacci Tableaux and Consecutive Pattern Avoidance

Observation

The set of column reading words for standard Fibonacci tableaux of size n is $\text{Av}_n(321, 312)$, the set of permutations of length n that avoid the consecutive patterns 312 and 321.

The Inversion Polynomial on $Av_n(321, 312)$

Theorem (C-K, 2013)

Let $\Pi = \{321, 312\}$. Then $l_0(\Pi; q) = l_1(\Pi; q) = 1$ and for $n \geq 2$,

$$l_n(\Pi; q) = l_{n-1}(\Pi; q) + (q + q^2 + \cdots + q^{n-1})l_{n-2}(\Pi; q).$$

Let ν be a Fibonacci shape and define

$$l_\nu(q) := \sum_{T \text{ shape } \nu} q^{\text{inv}(T)}$$

Note that

$$\sum_{|\nu|=n} l_\nu(q) = \sum_{\nu=1\mu} l_\nu(q) + \sum_{\nu=2\mu} l_\nu(q)$$

The Inversion Polynomial on $Av_n(321, 312)$

- $I_{1\mu}(q) = I_\mu(q)$ where $|\mu| = n - 1$

$$\text{inv} \left(\begin{array}{ccccc} & \bullet & \bullet & & \bullet \\ n & \bullet & \bullet & \bullet & \bullet \end{array} \right) = \text{inv} \left(\begin{array}{ccccc} \bullet & \bullet & & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \right)$$

- $I_{2\mu}(q) = (q + q^2 + \dots + q^{|\mu|+1}) I_\mu(q)$ where $|\mu| = n - 2$.

$$\text{inv} \left(\begin{array}{ccccc} k & \bullet & \bullet & & \bullet \\ n & \bullet & \bullet & \bullet & \bullet \end{array} \right) = n - k + \text{inv} \left(\begin{array}{ccccc} \bullet & \bullet & & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \right)$$

where $k = 1, \dots, n - 1$

The Inversion Polynomial on $Av_n(321, 312)$

Hence, for $\Pi = \{321, 312\}$

$$\begin{aligned} I_n(\Pi; q) &= \sum_{|\mu|=n-1} I_\mu(q) + \sum_{|\mu|=n-2} \left(q + q^2 + \cdots + q^{|\mu|+1} \right) I_\mu(q) \\ &= I_{n-1}(\Pi; q) + (q + q^2 + \cdots + q^{n-1}) I_{n-2}(\Pi; q) \end{aligned}$$

Inversion Polynomials for Consecutive Pattern Avoiding Permutations

In our recent paper, we compute the inversion polynomials for all but one set of permutations that simultaneously avoid a set of three or more consecutive patterns.

Class 5 - $\Pi = \{321, 312, 213, 132\}$

Theorem

$$I_n(\Pi; q) = 1 + q^2 + \cdots + q^{n-1}$$

In this case, $\text{Av}_n(\Pi)$ is in correspondence with standard Fibonacci tableaux that have at most one column of height two which, if it exists, must be the first column of the Fibonacci tableau.

If the Fibonacci tableau has all columns of height one then it corresponds with the permutation $\pi = 123 \cdots n$ which has an inversion number of 0.

If the Fibonacci tableau begins with a column of height two then the entry in the top row can be one of the numbers $1, 2, \dots, n-2$ (not $n-1$ since the corresponding permutation must avoid 132). If k is the number in the top row then the corresponding permutation has an inversion statistic of $n-k$. Summing over all possible k gives the result.

Class 7 - $\Pi = \{321, 312, 231\}$

Theorem

$$I_n(\Pi; q) = I_{n-1}(\Pi; q) + \sum_{k=1}^{\lfloor n/2 \rfloor} q^{k-1} C_{k-1}(q) \cdot I_{n-2k+1}(\Pi; q)$$

$\text{Av}_n(\Pi)$ is in one-to-one correspondence with standard Fibonacci tableaux for which the elements in the top of each column must decrease from left to right. (Note: the element in a column of height one is both in the top row of its column and in the bottom row of its column.)

$$T = \begin{array}{ccccccc} & e & c & & \bullet & & \\ \bullet & d & b & a & \bullet & \bullet & \end{array}$$

$w_c(T) = \bullet \bullet \bullet abcde \bullet$. Since abc must avoid 231, $c > a$. Since cde must also avoid 231, $e > c$.

Let $I_n^j(\Pi; q)$ denote the inversion polynomial for Fibonacci tableaux of size n whose shape starts with a j , where $j = 1, 2$.

If the Fibonacci tableau begins with a column of height one, then there must be an n in this column and n appears as the last element in the word of the tableau. Removing this n will not change the inversion statistic for this permutation and will give a tableau of size $n - 1$ with the given restrictions. Therefore,

$$I_n^1(\Pi; q) = I_{n-1}(\Pi; q).$$

If the Fibonacci tableau begins with a column of height two, let k be the smallest integer such that the first $k - 1$ columns have height two and the k th column has height one.

Since the elements in the top row of each column and the bottom row of each column must decrease from left to right, the numbers n , $n - 1$, $n - 2$, $\dots$, $n - 2k + 3$ must be in the first $k - 1$ columns and $n - 2k + 2$ must be in the k th column (of height 1). The tableaux formed from the remaining columns to the right of column k corresponds to a permutation of size $n - 2(k - 1) - 1 = n - 2k + 1$.

For example, for $n = 15$ and $k = 4$ we have

$$T = \begin{array}{cccccccc} 13 & 11 & 10 & & 6 & 5 & & 1 \\ 15 & 14 & 12 & 9 & 8 & 7 & 4 & 3 & 2 \end{array}$$

$$T = \begin{array}{ccc|c|ccc} 13 & 11 & 10 & & 6 & 5 & 1 \\ 15 & 14 & 12 & 9 & 8 & 7 & 4 & 3 & 2 \end{array}$$

Since $n - 2k + 2$ is in column k , removal of this element does not change the inversion statistic and also gives a tableau in the first $k - 1$ columns that corresponds (with relabeling) to a tableau of size $2(k - 1)$ with all columns of height two whose elements in the top row of each column decrease from left to right.

$$\begin{array}{ccc} 13 & 11 & 10 \\ 15 & 14 & 12 \end{array} \rightarrow \begin{array}{ccc} 4 & 2 & 1 \\ 6 & 5 & 3 \end{array} \rightarrow \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}$$

These tableaux are counted by the Catalan number C_{k-1} and the inversion polynomial on such tableau is given by $q^{k-1}C_{k-1}(q)$ where $C_{k-1}(q)$ are the q -Catalan polynomials. This gives us the inversion polynomial

$$q^{k-1}C_{k-1}(\Pi; q)I_{n-2k+1}(\Pi; q),$$

and we have

$$I_n^2(\Pi; q) = \sum_{k=1}^{\lfloor n/2 \rfloor} q^{k-1}C_{k-1}(q) \cdot I_{n-2k+1}(\Pi; q).$$

Thus, with initial conditions

$$I_2^1(\Pi; q) = 1 \text{ and } I_2^2(\Pi; q) = I_3^2(\Pi; q) = q,$$

we have

$$I_n(\Pi; q) = I_{n-1}(\Pi; q) + \sum_{k=1}^{\lfloor n/2 \rfloor} q^{k-1} C_{k-1}(q) \cdot I_{n-2k+1}(\Pi; q)$$

Further Research

- We have determined the inversion polynomial for (almost) all Π where Π is a collection of three or more consecutive patterns. We continue to work on the remaining class of three consecutive patterns and the remaining classes of two consecutive patterns.
- If $\{321, 132\}$ is not a subset of Π , we use a more general notion called a **strip tableaux** to model the permutations in $Av_n(\Pi)$.
- We are also working on mixed patterns (some consecutive letters and some not) and on how we can use our strip shape model to count these classes.
- Killpatrick's summer undergraduate research students are investigating consecutive patterns of length four.

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