Deflatability of permutation classes

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Joint work with
Mike Atkinson (UO), Cheyne Homberger (UF) and Jay Pantone (UF).

Permutation Patterns 2014, Johnson City

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The definitions

A *permutation* is a set of points in the plane (no two on a horizontal or vertical line)

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The definitions

I lied, it's the equivalence class of such a set of points under arbitrary vertical and horizontal rescaling.

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The definitions

A *(permutation) class* is a set of permutations closed under erasing points.

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The definitions

A permutation is *simple* if no box contains at least two points and is not cut horizontally or vertically by an exterior point unless it contains all the points.

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The definitions

An *interval* in a permutation is a box (or the elements it contains) that is not cut horizontally or vertically by any point not in it.

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The definitions

So a permutation is *simple* if its only intervals contain 0, 1 or all the points (these are *trivial* intervals).

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The definitions

A permutation class is *deflatable* if it has a proper subclass containing the same simple permutations.

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The definitions

Equivalently, a class $\mathcal{C}$ is *not deflatable* if every permutation in $\mathcal{C}$ has a simple extension in $\mathcal{C}$.

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What's the point of deflatability?

- If a class $\mathcal{C}$ is deflatable, then the class $\mathcal{D}$ formed by closing its simples under subpermutation is strictly smaller.
- Understand $\mathcal{D}$.
- Understand the allowed inflations to get $\mathcal{C}$.

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The question (and answer)

Question: Which principal permutation classes $\mathrm{Av}(\pi)$ are deflatable?

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Answer: It's complicated.

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The question (and answer)

Question: Which principal permutation classes $\mathrm{Av}(\pi)$ are deflatable?

Answer: It's complicated, and we don't know the whole story.

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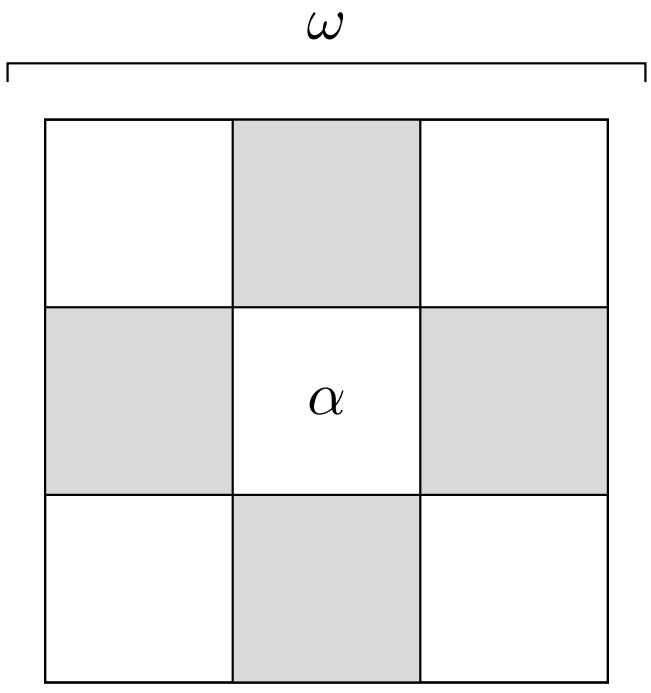
Intervals and extensions



$$2 \leq |\alpha| < |\omega|$$

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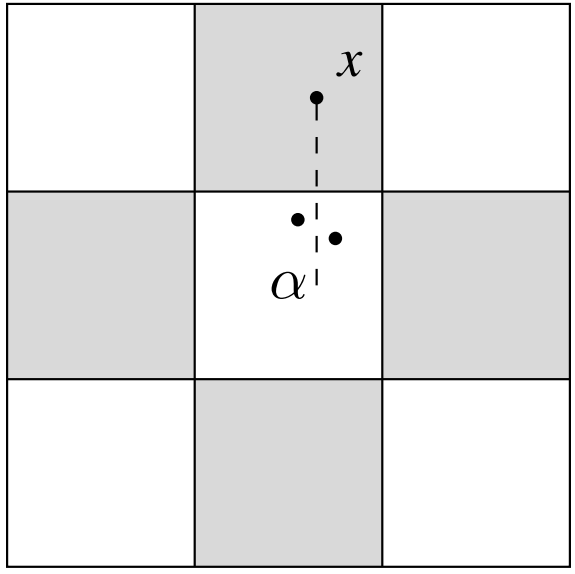

x cuts α

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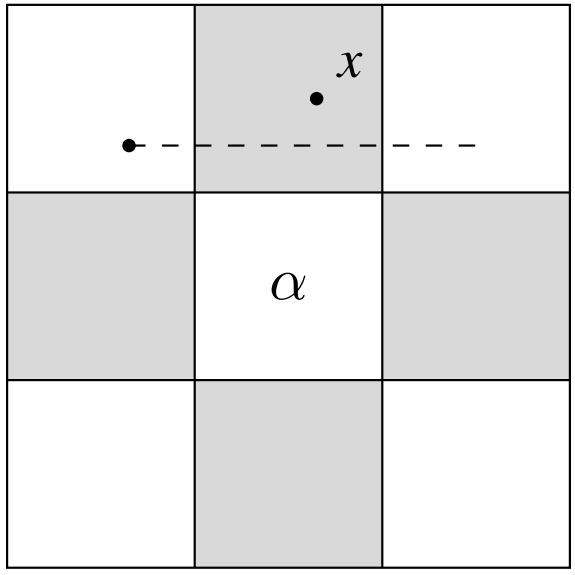
Intervals and extensions



x is separated from α

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An observation

Let ω be a permutation containing an interval α and suppose that $\omega' = \omega \cup \{x\}$ where x cuts and is separated from α. If x belongs to a proper interval of ω' then at least one of the following must hold:

- ω is decomposable,
- α is not a maximal interval of ω.

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If neither holds:

- ω' is indecomposable,
- any proper interval of ω' is a proper interval of ω, and
- α is not an interval of ω'.

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A plan

Let $|\pi| \geq 4$. To show $\mathrm{Av}(\pi)$ is not deflatable:

1. Show that every permutation of $\mathrm{Av}(\pi)$ is contained in an indecomposable one.
2. Show that if α is a maximal interval of a non-simple element, ω, of $\mathrm{Av}(\pi)$ then there is an extension $\omega' = \omega \cup \{x\}$ also in $\mathrm{Av}(\pi)$ where α is cut by and separated from x.

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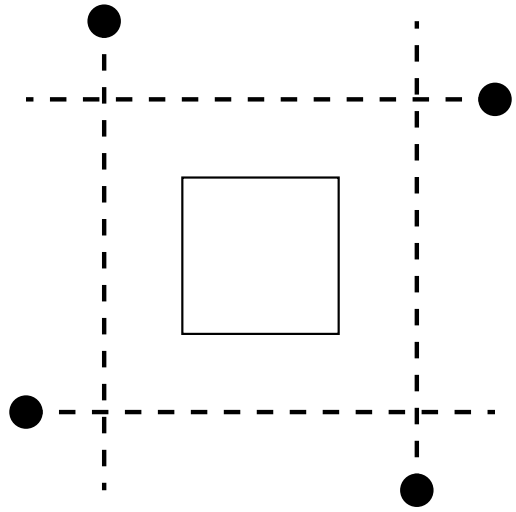
Av(2143): Embedding in an indecomposable



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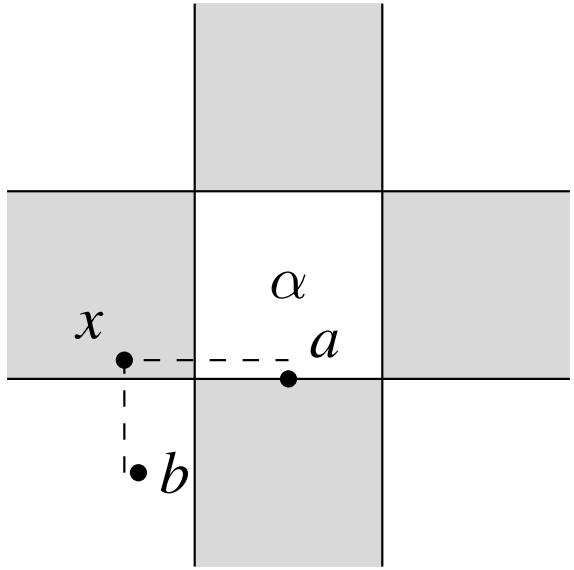
Av(2143): Breaking an interval (I)

Suppose there is a 21 below and left of the interval α. Let b be the 1 of the leftmost such 21. Let $a = \min \alpha$ and add x just above a and just to the left of b. Check that it can't make a 2143.



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Av(2143): Breaking an interval (II)

- We can assume the area below and left of α is increasing.

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- We can assume the area below and left of α is increasing.
- To avoid decomposability, above left or below right must be non empty (say the former by symmetry).

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$\mathrm{Av}(2143)$: Breaking an interval (II)

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- To avoid decomposability, above left or below right must be non empty (say the former by symmetry).
- Now let b be the rightmost element of that area.

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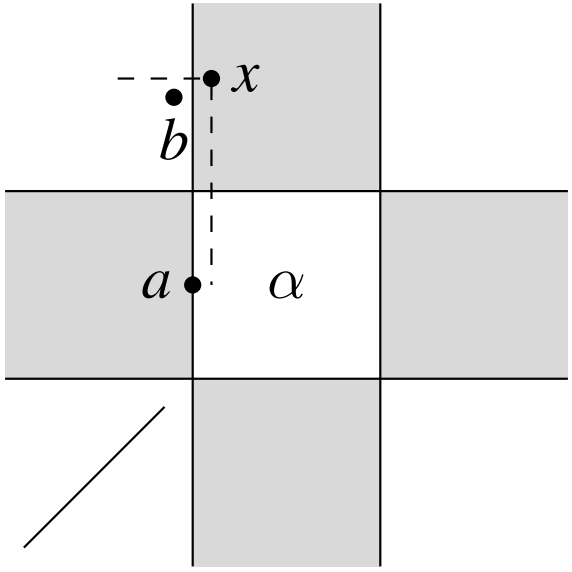

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Towards part 2 of the argument

- As my Olympiad students would say “case bashing”.

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- Try adding “cutting and separated” x in obvious places (near the edge of α, separated by at most one element etc.)

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Theorem

If $\pi = \lambda \oplus \rho$ and $|\lambda|, |\rho| \geq 2$ then $\mathrm{Av}(\pi)$ is not deflatable.

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More on the decomposable case

A *bond* is an interval of size 2. Bonds can be increasing or decreasing.

Theorem

If $\pi = 1 \oplus \rho$ and ρ does not contain both an increasing and a decreasing bond, then $\mathrm{Av}(\pi)$ is not deflatable.

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More on the indecomposable case

Ummm . . .

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More on the indecomposable case

Ummm . . .

Theorem (MA + VV)

$\mathrm{Av}(2413)$ *is not deflatable.*

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Deflatable examples

To show that $\mathrm{Av}(\pi)$ is deflatable one needs to find some $\omega \in \mathrm{Av}(\pi)$ without a simple extension. This is greatly facilitated by:

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Which is most easily illustrated using *PermLab*.

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Remaining issues

- The indecomposable case, in particular is deflatability, non-deflatability, or “it depends” typical?
- Systematic construction of deflatable examples (we have an infinite set but all very *ad hoc*)
- In the principal case, do one point extensions suffice (to reduce the number of intervals)? They do not in general (see *PermLab* again).

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Thank you!

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