

Convexity Invariants of the Hoop Closure on Permutations

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Eliakim Hastings Moore — father of closure theory (1910).

The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.

Looking for similarities in various notions of closure.

E. H. Moore has 31 students and 18,326 descendants.

Here is what the internet pundits have to say about closure:

It's crucial that you try to find closure.

—— Joshua Duvauchelle

Any useful discussion about achieving “closure” . . .

should start with a solid working definition of the term “closure.”

—— R. G. Baldino, MSW, MCSW

It can be beneficial to arrange . . . for a conversation specifically about achieving closure.

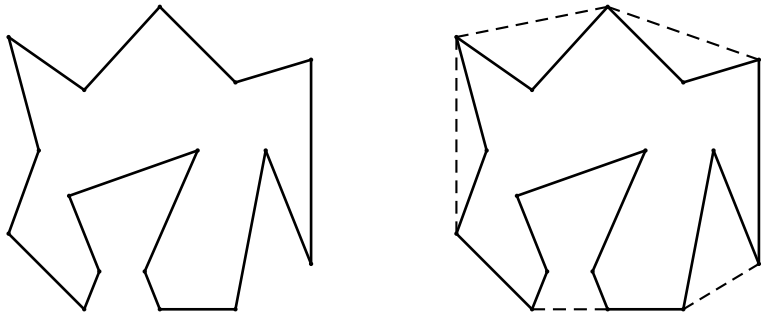
—— Joe Burnham

If anyone ever figured out how to bottle the ability to achieve closure . . ., they would be rich and famous.

No matter what the situation, closure . . . comes from within.

Closure takes time. Unfortunately, time is something we have no control over.

Figure: Convex Closure in the Plane.



CLOSURE: Any process that takes some starting situation and completes it by filling in essential parts that are missing.

A *closure space* $(X, \mathcal{C})$ consists of a *ground set* X and a family $\mathcal{C}$ of *closed* subsets of X such that

- the whole space X is closed, and
- the arbitrary intersection of closed sets is closed.

Closure from outside

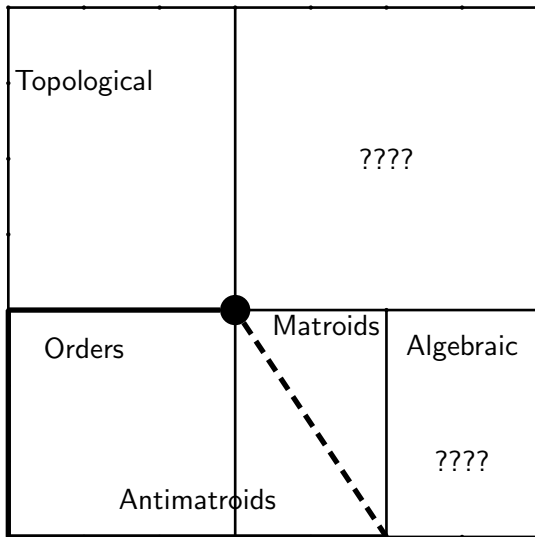
The intersection of all closed sets containing a subset S of X is the *closure* of S .

$\mathcal{C}(S)$ denotes the closure of S with respect to the family $\mathcal{C}$.

Closure from within

A set C is *closed* provided C contains all points dependent on it.

Figure: Map of the Land of Closure



Some of this material will be very familiar

Some of it will not.

I want to be meticulous in connecting the two.

The material can be looked at from a number of different angles.

Let's start by looking at three ways one can view a permutation.

Permutation π as a string of n distinct integers $1, \dots, n$.

For each index j choose a set of allowed values for the j th term.
This is the j th *hoop*.

A choice of one hoop H_j for each index j is a *hoop system*.

A permutation $\pi(1), \dots, \pi(n)$ *jumps through* the hoops H_j
of a hoop system H iff $\pi(j) \in H_j$ for all j .

hoops	$\{2, 4, 5\}$	$\{1, 2, 3\}$	$\{1, 5\}$	$\{1, 5\}$	$\{3\}$
permutation	4	2	1	5	3

Such a permutation can be regarded as a *system of distinct representatives* (SDR) for the family of hoops.

Table: Permutation as a collection of cells in a matrix

	α				
		α			
					α
			α		
				α	
α					

α is the name of the permutation.

If we use 1 instead of α , we get a *permutation matrix*.

Table: Hoop system as a template of cells in a matrix

Hoops	H_1	H_2	H_3	H_4	H_5	H_6
indices/	$\{1, 6\}$	$\{1, 2, 5\}$	$\{2, 3, 4\}$	$\{3, 4\}$	$\{3, 5\}$	$\{3, 6\}$
1	*	*				
2		*	*			
3			*	*	*	*
4			*	*		
5		*			*	
6	*					*

Table: Permutations $\alpha, \beta, \gamma, \delta$ fitting the template.

$\beta\gamma\delta$	α				
	$\gamma\delta$	$\alpha\beta$			
		δ	γ	β	α
		γ	$\alpha\beta\delta$		
	β			$\alpha\gamma\delta$	
α					$\beta\gamma\delta$

Table: Permutation as an injective function

4					x
3		x			
2				x	
1	x				
0			x		
	0	1	2	3	4

The function $x \longrightarrow 2x + 1$ in $\mathbb{Z}_5$.

Closure from outside

For each hoop system H , let $\Pi(H)$ denote the set of all permutations that jump through the hoops of H .

These sets form the closed sets of a closure system $\mathcal{H}$ on the permutations of $1, \dots, n$.

We also have a reverse process:

For any set S of permutations, consider their *carrier* $\Psi(S)$ — the set of all cells covered by the permutations in S .

The closure operator is given by $S \longrightarrow \Pi(\Psi(S))$ where S is a set of permutations.

A permutation π is *dependent* on a set S of permutations iff every cell of π is a cell of some permutation in S .

Closure from within

A set C of permutations is *closed* provided C contains all permutations dependent on it.

Table: In the closure of three parallels

1	0	1		$\alpha, 3$			
2		$\alpha, 0$	1		3		
3			0	1		$\alpha, 3$	
4				0	$\alpha, 1$		3
5	$\alpha, 3$				0	1	
6		3				0	$\alpha, 1$
7	1		$\alpha, 3$				0

The three permutations

$x \rightarrow x + 0$, $x \rightarrow x + 1$, $x \rightarrow x + 3 \pmod{7}$

have a carrier that contains the permutation α .

A set I in a closure space is *independent* provided $p \notin \mathcal{C}(I \setminus \{p\})$ for all $p \in I$.

Let π be a permutation in a set S of permutations.
A cell z is a *private cell* for π with respect to S
provided z is a cell of π but of no other permutation in S .
The set S is independent in hoop closure iff
every permutation in S has a private cell.

Table: The permutations $\alpha, \beta, \gamma, \delta$ again.

$\beta\gamma\delta$	α				
	$\gamma\delta$	$\alpha\beta$			
		δ	γ	β	α
		γ	$\alpha\beta\delta$		
	β			$\alpha\gamma\delta$	
α					$\beta\gamma\delta$

Using an affine plane of prime power order q .

Let q be a prime power.

Coordinatize the cells of a $q \times q$ matrix
by elements from the finite field $GF(q)$.

Analytic geometry works over $GF(q)$ in analogy to the real case.

Lines are defined by equations of the form

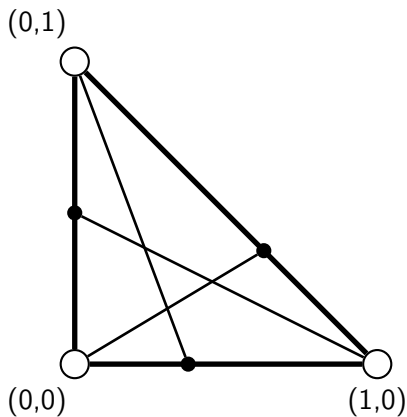
- ▶ $y = mx + b$ where $m \neq 0$. These are the *useful* lines.
- ▶ $y = b$. These are the horizontal lines.
- ▶ $x = a$. These are the vertical lines.

The graphs of the useful lines form permutations.

The graphs of vertical and horizontal lines do not yield permutations.

From each vertex, take the line to each of the $q - 2$ “interior” points on the opposite side.

Figure: Independent set of $3q - 6$ permutations.



A point p is an extremepoint of a closed set K provided $K \setminus \{p\}$ is also closed.

A set S in a closure space is *free* if all subsets of S are closed.

This is equivalent to saying that

- S is closed and independent.

Also equivalent to saying that

- S is closed and all points of S are extremepoints.

private cell of a permutation π in a set S of permutations
 z is a cell of π but of no other permutation in S .

monic cell of a set C of cells

there is a unique permutation in C containing the cell z .

If a permutation π contains a monic cell
for the carrier of a set S of permutations,
then that cell is a private cell for π in S .

THM. π is an extremepoint of a closed set K of permutations iff
 π contains a monic cell with respect to K .

THM. A set S of permutations is free iff
every permutation in S contains a monic cell with respect to S .

Note that in the above THM S is not assumed to be closed.
That is a consequence.

Table: The permutations $\alpha, \beta, \gamma, \delta$ again.

$\beta\gamma\delta$	α				
	$\gamma\delta$	$\alpha\beta$			
		D	γ	β	α
		γ	$\alpha\beta\delta$		
	β			$\alpha\gamma\delta$	
α					$\beta\gamma\delta$

Table: The permutations $\alpha, \beta, \gamma, \delta$ and again.

$\beta\gamma\delta$	α				
	$\gamma\delta$	x			
		D	x	x	x
		x	$\alpha\beta\delta$		
	β			$\alpha\gamma\delta$	
α					$\beta\gamma\delta$

Table: The permutations $\alpha, \beta, \gamma, \delta$ again and again.

$\beta\gamma\delta$	α				
	D_2	x			
		D	x	x	x
		x	D_2		
	β			D_2	
α					D_2

Table: The permutations $\alpha, \beta, \gamma, \delta$ again and again.

$\beta\gamma\delta$	x				
	D_2	x			
		D	x	x	x
		x	D_2		
	x			D_2	
x					D_2

Table: The permutations $\alpha, \beta, \gamma, \delta$ again and again.

D_3	x				
	D_2	x			
		D	x	x	x
		x	D_2		
	x			D_2	
x					D_2

Here the permutations $x + 0, x + 1, x + 3$ have private cells and are independent. α is dependent on these three.

Table: Monic cells and private cells.

0	1		$\alpha, 3$			
	$\alpha, 0$	1		3		
		0	1		$\alpha, 3$	
			0	$\alpha, 1$		3
$\alpha, 3$				0	1	
	3				0	$\alpha, 1$
1		$\alpha, 3$				0

There are no monic cells, however.

Each cell of the carrier belongs to one of $x + 0, x + 1, x + 3$ and some shift of α .

John Dee in his 1570 translation of Euclid takes the trouble to point out that
Every certayne Line, hath two endes: The endes of a line, are Pointes called.

The closure of a pair of points is called the *segment* between them.

We will now look at segments in hoop closure and at *pairwise free* sets:

the segment between each pair of points consists of just those points.

If α and β are permutations, a square set B of cells in the template for α and β is a *box* for α and β provided α and β both pass through B . Here is an example.

Table: Box decomposition and flip-flopping

α	β					
β	α					
		α, β				
			α	β		
				α		β
					β	α
			β		α	

Box Decomposition

α	β					
β	α					
		α, β				
			β	α		
				β		α
					α	β
			α		β	

Flip-flopping

A box is specified by a pair of sets of indices (I, J) where I is a set of row indices and J is a set of column indices.

This is a box for permutations α and β provided

- $\alpha(J) = I = \beta(J)$.

Consider $j \in J$. Let $i = \beta(j)$. Then $i \in I$

Since $\alpha(J) = I$, there is a $k \in J$ with $\alpha(k) = i = \beta(j)$.

Applying α^{-1} , we get

$$k = \alpha^{-1}\beta(j).$$

That is, J is invariant under the action of $\alpha^{-1}\beta$.

Thus the boxes for the pair α and β

correspond to the cycles in the cycle decomposition of $\alpha^{-1}\beta$.

The Size of a Segment

THM: The number of permutations in the segment between two permutations α and β is given by

2^m where m is the number of cycles of length > 1 in the cycle decomposition of $\alpha^{-1}\beta$.

Note that this is the same number if we reverse α and β since the inverse $\beta^{-1}\alpha$ of $\alpha^{-1}\beta$ has the same cycle structure as $\alpha^{-1}\beta$.

THUS a set S of permutations is pairwise free iff for all $\alpha, \beta \in S$, the permutation $\alpha^{-1}\beta$ has at most one cycle of length > 1 .

Table: The permutations $\alpha, \beta, \gamma, \delta$ one last time.

$\beta\gamma\delta$	α				
	$\gamma\delta$	$\alpha\beta$			
		δ	γ	β	α
		γ	$\alpha\beta\delta$		
	β			$\alpha\gamma\delta$	
α					$\beta\gamma\delta$

$$\delta\alpha = (1632)(4)(5)$$

$$\delta\beta = (1)(253)(4)(6)$$

$$\delta\gamma = (1)(2)(34)(5)(6)$$

Consider a Cayley graph for the symmetric group on n symbols where the generators are all individual cycles of length > 1

The pairwise free set correspond to the cliques in this graph.

Thank you for your attention.

