An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

Jul. 11, 2014

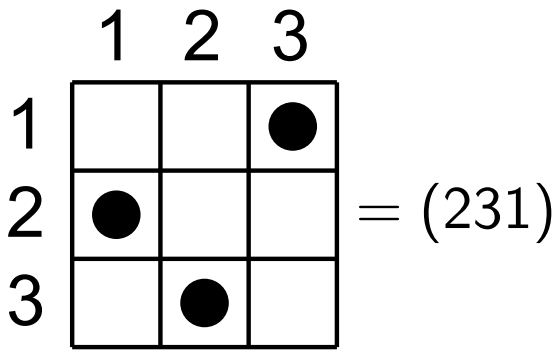
Pattern avoidance in permutations

New result

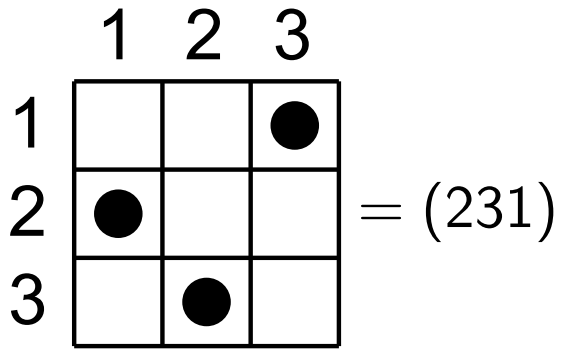
Shape-Wilf-equivalence

Shape-inv-Wilf-equivalence

Even-Wilf-equivalence



$= (231)$



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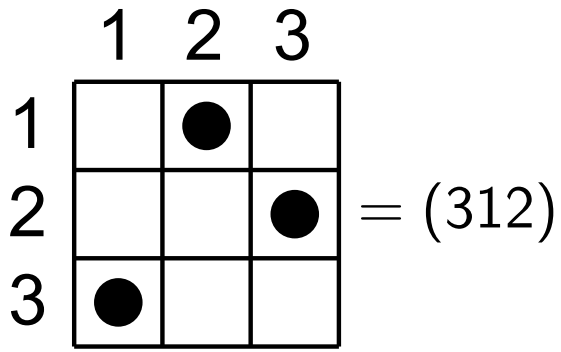
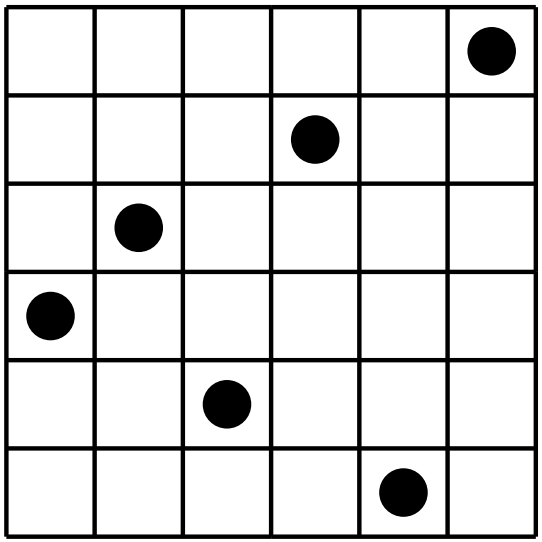
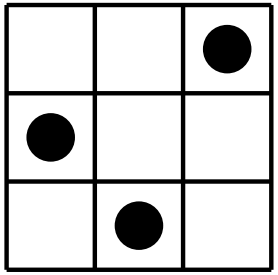


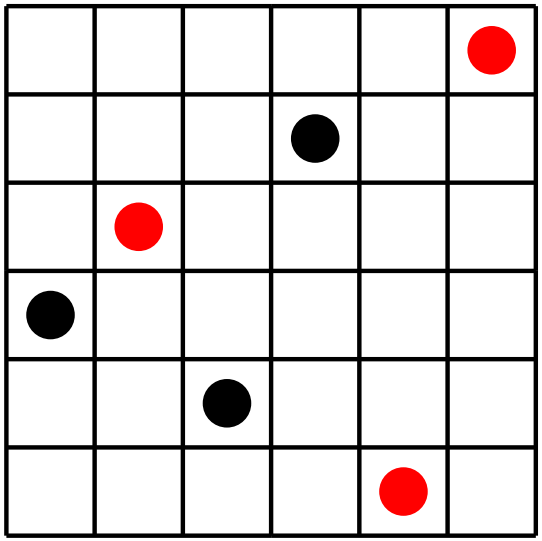
$= (312)$



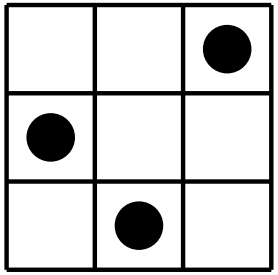
contains



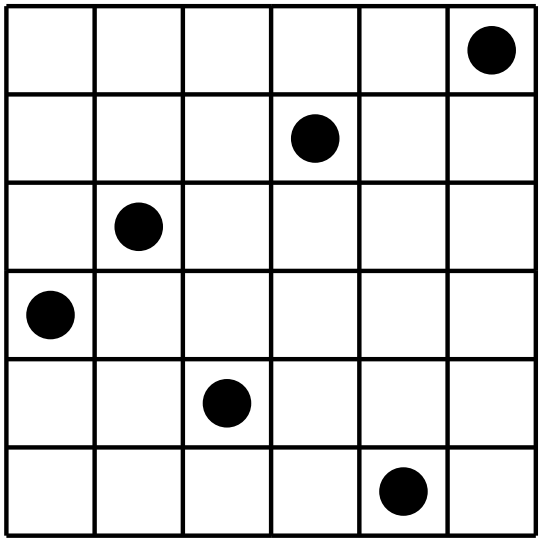
(435261) contains (231).



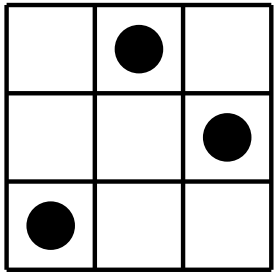
contains



(435261) contains (231).



avoids



(435261) avoids (312).

$S_n(\pi)$: Set of permutations of length n that avoid π.

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$S_4(231)$			$S_4(321)$		
(1234)	(1243)	(1324)	(1234)	(1243)	(1324)
(1423)	(1432)	(2134)	(1342)	(1423)	(2134)
(2143)	(3124)	(3214)	(2143)	(2314)	(2341)
(4123)	(4132)	(4213)	(2413)	(3124)	(3142)
(4312)	(4321)		(3412)	(4123)	

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$S_4(321)$		
(1234)	(1243)	(1324)
(1342)	(1423)	(2134)
(2143)	(2314)	(2341)
(2413)	(3124)	(3142)
(3412)	(4123)	

π	$\lvert S_3(\pi)\rvert$	$\lvert S_4(\pi)\rvert$	$\lvert S_5(\pi)\rvert$	$\lvert S_6(\pi)\rvert$	$\lvert S_7(\pi)\rvert$	...
(231)	5	14	42	132	429	...

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(4123)	(4132)	(4213)
(4312)	(4321)	

$S_4(321)$		
(1234)	(1243)	(1324)
(1342)	(1423)	(2134)
(2143)	(2314)	(2341)
(2413)	(3124)	(3142)
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$S_4(231)$			$S_4(321)$		
(1234)	(1243)	(1324)	(1234)	(1243)	(1324)
(1423)	(1432)	(2134)	(1342)	(1423)	(2134)
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π	$\lvert S_3(\pi)\rvert$	$\lvert S_4(\pi)\rvert$	$\lvert S_5(\pi)\rvert$	$\lvert S_6(\pi)\rvert$	$\lvert S_7(\pi)\rvert$	...
(231)	5	14	42	132	429	...
(321)	5	14	42	132	429	...

(231) and (321) are **Wilf-equivalent**: $|S_n(231)| = |S_n(321)|$ for all n.

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Herbert S. Wilf (1931–2012)

- In 2012, T. Dokos, T. Dwyer, B.P. Johnson, B.E. Sagan, and K. Selsor studied a strengthening of Wilf-equivalence involving the use of permutation statistics (functions from the set of permutations to some set such as the non-negative integers.)

- In 2012, T. Dokos, T. Dwyer, B.P. Johnson, B.E. Sagan, and K. Selsor studied a strengthening of Wilf-equivalence involving the use of permutation statistics (functions from the set of permutations to some set such as the non-negative integers.)
- One such permutation statistic is inversion number.

Inversion number of a permutation: We count each pair of positions where the value decreases from left to right.

(3142)

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(3142)

Count: |

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Inversion number of a permutation: We count each pair of positions where the value decreases from left to right.

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Count: | | |

$\mathrm{inv}(3142) = 3$

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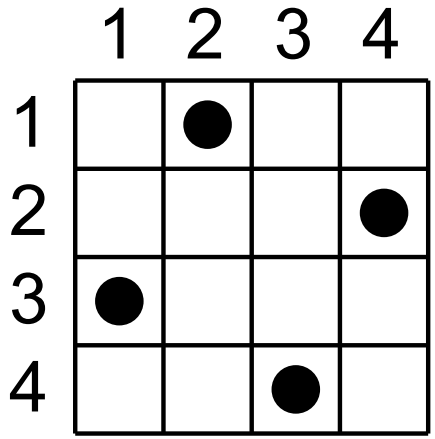
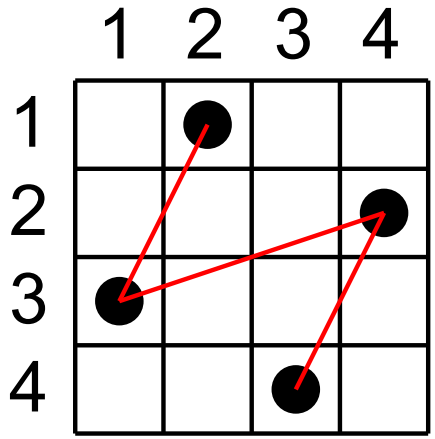
$\mathrm{inv}(3142) = 3$



$\mathrm{inv}(3142) = 3$



$S_4(231)$			$S_4(321)$		
(1234)	(1243)	(1324)	(1234)	(1243)	(1324)
(1423)	(1432)	(2134)	(1342)	(1423)	(2134)
(2143)	(3124)	(3214)	(2143)	(2314)	(2341)
(4123)	(4132)	(4213)	(2413)	(3124)	(3142)
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$S_4(231)$			$S_4(321)$		
(1234)	(1243)	(1324)	(1234)	(1243)	(1324)
(1423)	(1432)	(2134)	(1342)	(1423)	(2134)
(2143)	(3124)	(3214)	(2143)	(2314)	(2341)
(4123)	(4132)	(4213)	(2413)	(3124)	(3142)
(4312)	(4321)		(3412)	(4123)	

Group each of the permutations by inversion number.

$S_4(231)$	inv	$S_4(321)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (1423) (2143) (2314) (3124)
(1432) (3214) (4123)	3	(2341) (2413) (3142) (4123)
(4132) (4213)	4	(3412)
(4312)	5	
(4321)	6	

$S_4(231)$	inv	$S_4(321)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (1423) (2143) (2314) (3124)
(1432) (3214) (4123)	3	(2341) (2413) (3142) (4123)
(4132) (4213)	4	(3412)
(4312)	5	
(4321)	6	

$$I_4(231, q) = 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$$
$$I_4(321, q) = 1 + 3q + 5q^2 + 4q^3 + q^4$$

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$S_4(231)$	inv	$S_4(321)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (1423) (2143) (2314) (3124)
(1432) (3214) (4123)	3	(2341) (2413) (3142) (4123)
(4132) (4213)	4	(3412)
(4312)	5	
(4321)	6	

$I_4(231, q) = 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$
$I_4(321, q) = 1 + 3q + 5q^2 + 4q^3 + q^4$
(231) and (321) are not **inv-Wilf-equivalent**.

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$S_4(231)$	inv	$S_4(321)$
(1234)	0	
(1243) (1324) (2134)	1	
(1423) (2143) (3124)	2	
(1432) (3214) (4123)	3	
(4132) (4213)	4	
(4312)	5	
(4321)	6	

$S_4(231)$	inv	
(1234)	0	
(1243) (1324) (2134)	1	
(1423) (2143) (3124)	2	
(1432) (3214) (4123)	3	
(4132) (4213)	4	
(4312)	5	
(4321)	6	

$S_4(231)$	inv	$S_4(312)$
(1234)	0	
(1243) (1324) (2134)	1	
(1423) (2143) (3124)	2	
(1432) (3214) (4123)	3	
(4132) (4213)	4	
(4312)	5	
(4321)	6	

$S_4(231)$	inv	$S_4(312)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (2143) (2314)
(1432) (3214) (4123)	3	(1432) (3214) (2341)
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(4312)	5	(3421)
(4321)	6	(4321)

$S_4(231)$	inv	$S_4(312)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
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$$I_4(231, q) = 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$$
$$I_4(312, q) = 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$$

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(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
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$I_4(231, q) = 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$
$I_4(312, q) = 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$
Similarly, $I_n(231, q) = I_n(312, q)$ for all n.

$S_4(231)$	inv	$S_4(312)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (2143) (2314)
(1432) (3214) (4123)	3	(1432) (3214) (2341)
(4132) (4213)	4	(2431) (3241)
(4312)	5	(3421)
(4321)	6	(4321)

$I_4(231, q) = 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$
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Similarly, $I_n(231, q) = I_n(312, q)$ for all n.
(231) and (312) are **inv-Wilf-equivalent**.

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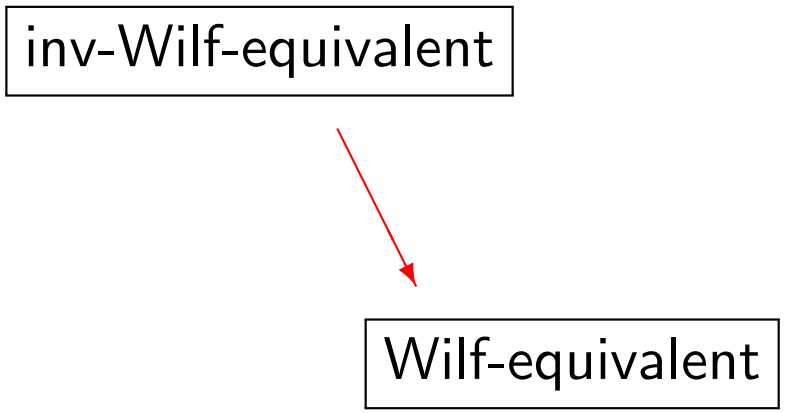
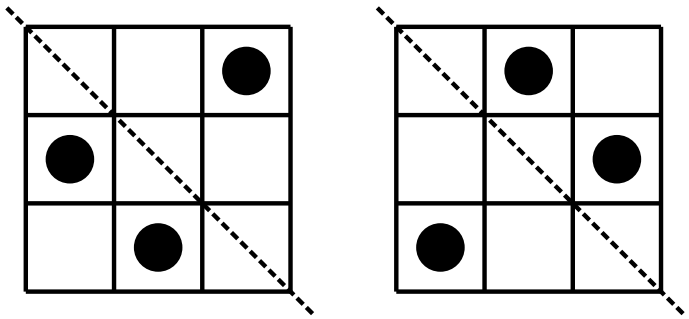
Similarly, $I_n(231, q) = I_n(312, q)$ for all n.
(231) and (312) are **inv-Wilf-equivalent**.
Inv-Wilf-equivalence implies Wilf-equivalence.

Wilf-equivalent


inv-Wilf-equivalent
Wilf-equivalent

In fact, inv-Wilf-equivalence of (231) and (312) is trivial.



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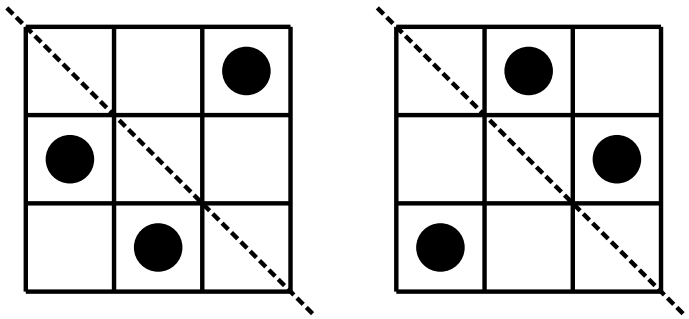

If a permutation avoids (231), then its diagonal reflection avoids (312), and has the same inversion number.

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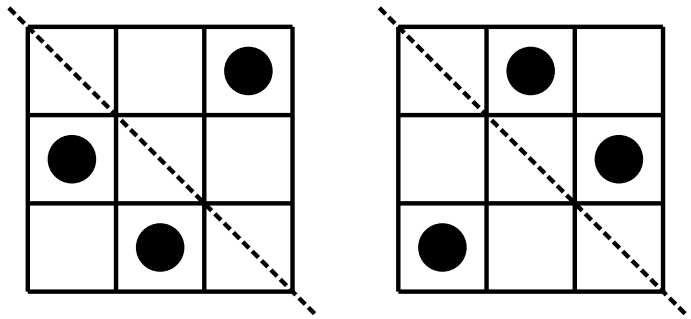

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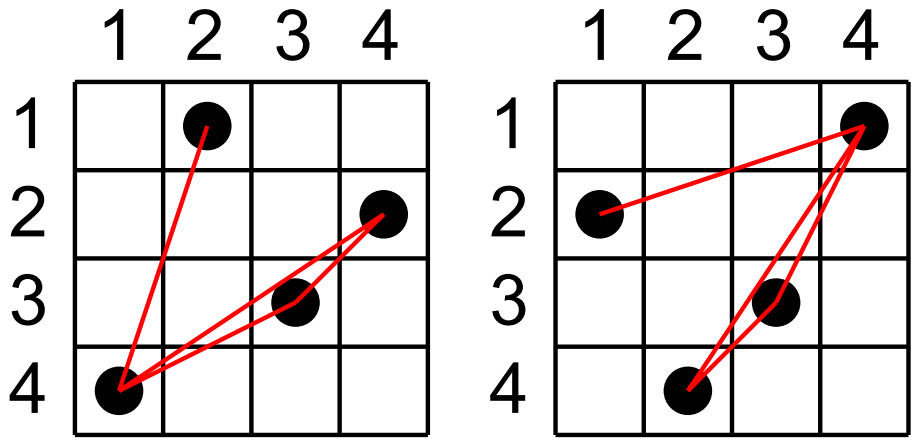

- Dokos et al. conjecture (2012) that all inv-Wilf-equivalences are trivial (one permutation can be obtained from the other by reflection through either diagonal or by 180-degree rotation).

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- Verified computationally in the same paper for all permutation pairs π, π' of length ≤ 5: Sufficient to show $I_n(\pi, q) \neq I_n(\pi', q)$ for a single value of n. (In this case, $n = 8$ suffices.)

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- Verified computationally in the same paper for all permutation pairs π, π' of length ≤ 5: Sufficient to show $I_n(\pi, q) \neq I_n(\pi', q)$ for a single value of n. (In this case, $n = 8$ suffices.)
- What about permutation pairs of length 6?

It turns out that (231564) and (312564) form a nontrivial inv-Wilf-equivalent pair!

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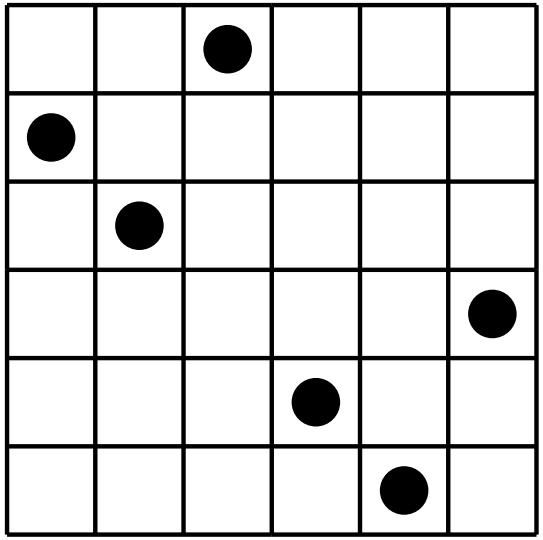
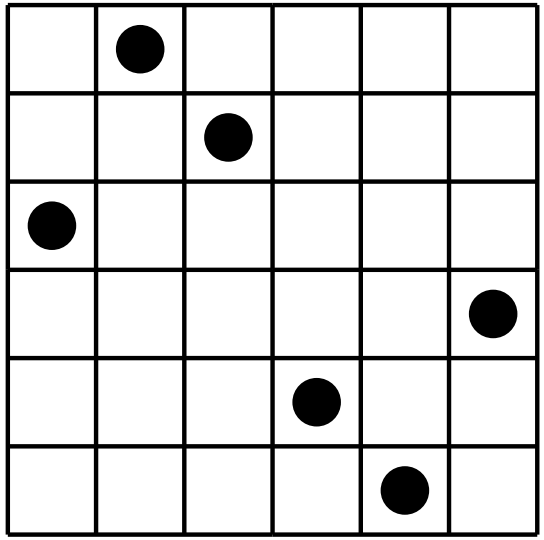
(231564)



$(231) \oplus (231)$

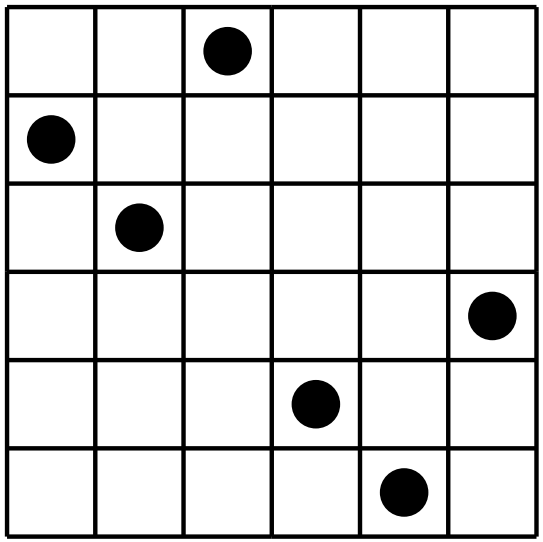
(312564)



$(312) \oplus (231)$

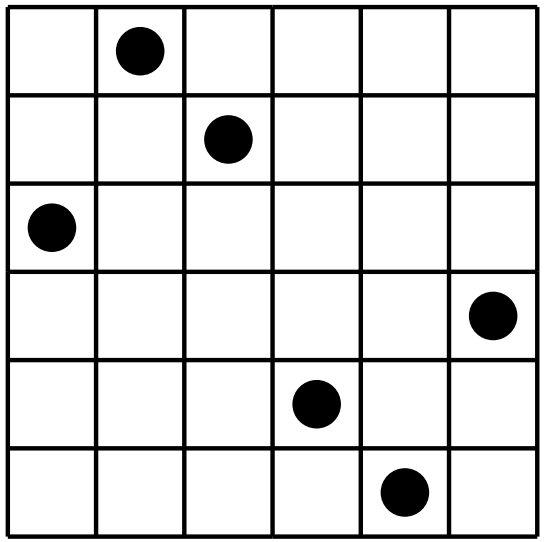
It turns out that (231564) and (312564) form a nontrivial inv-Wilf-equivalent pair!

(231564)



$(231) \oplus (231)$

(312564)

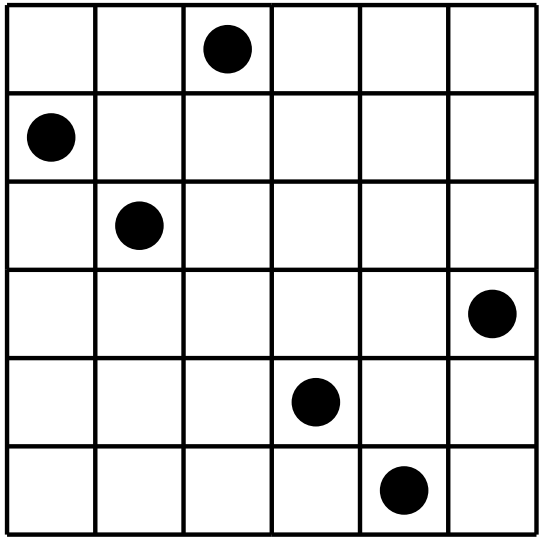


$(312) \oplus (231)$

In fact, for every permutation γ, the pair of permutations $(231) \oplus \gamma$ and $(312) \oplus \gamma$ is inv-Wilf-equivalent!

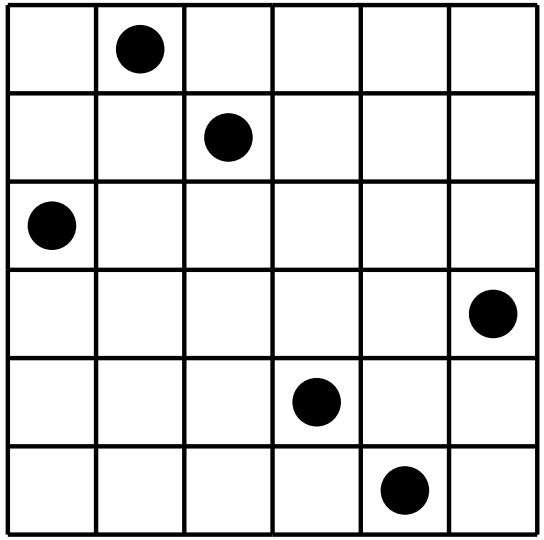
It turns out that (231564) and (312564) form a nontrivial inv-Wilf-equivalent pair!

(231564)



$(231) \oplus (231)$

(312564)

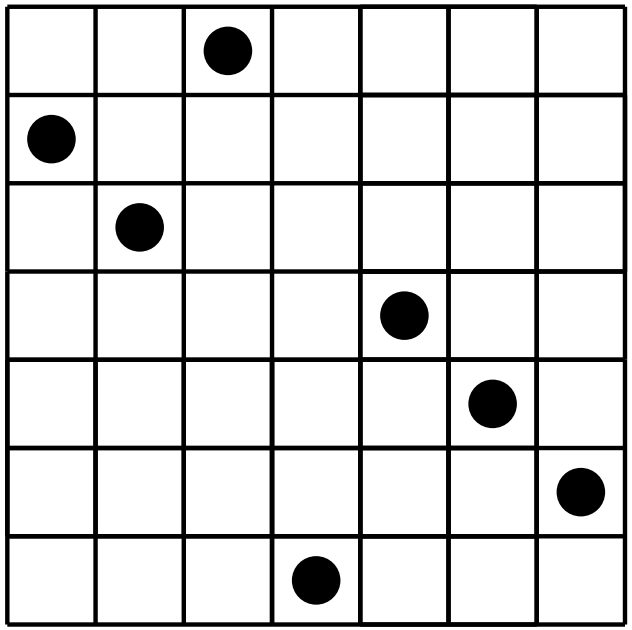


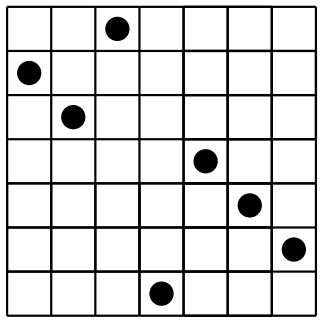
$(312) \oplus (231)$

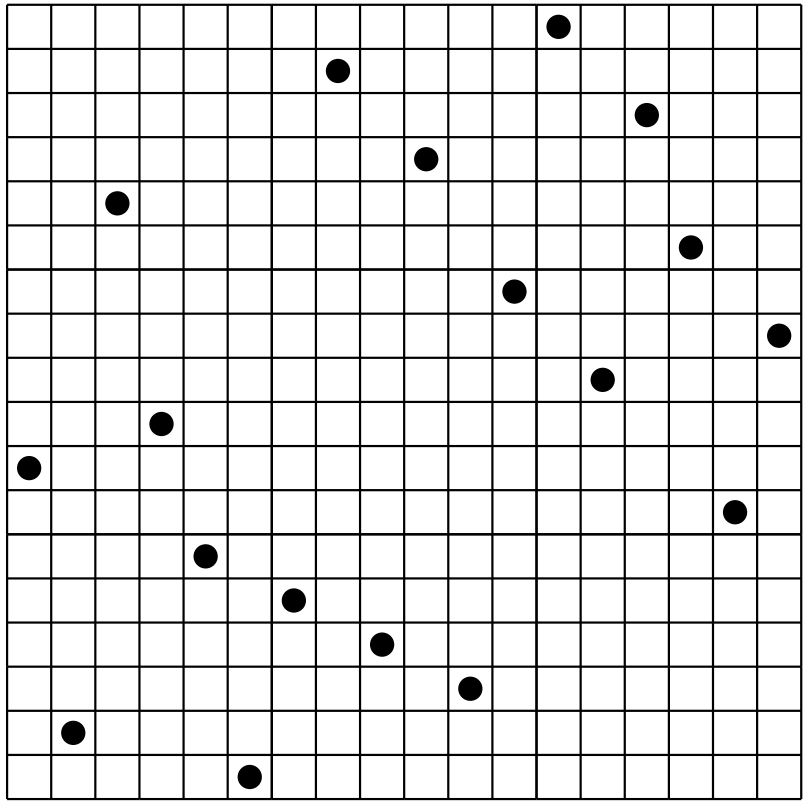
In fact, for every permutation γ, the pair of permutations $(231) \oplus \gamma$ and $(312) \oplus \gamma$ is inv-Wilf-equivalent!
This results in an infinite number of pairs of nontrivial inv-Wilf-equivalent permutations.

- What happens when a permutation avoids $(231) \oplus \gamma$?

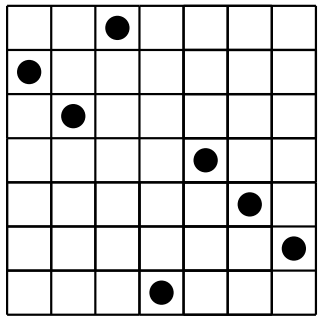
- What happens when a permutation avoids $(231) \oplus \gamma$?
- Let's take $(2317456) = (231) \oplus (4123)$ as an example.

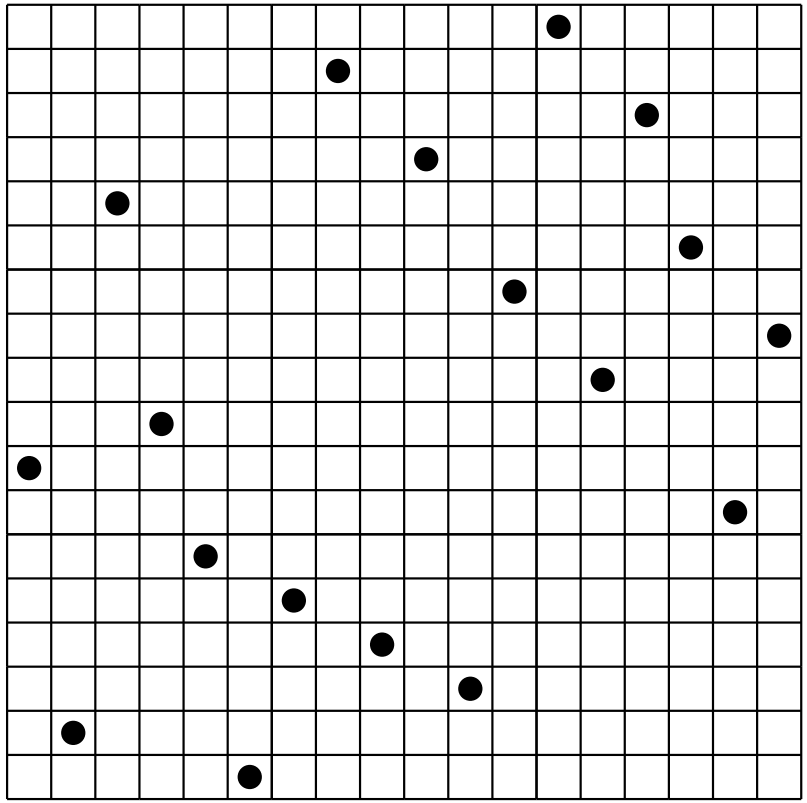




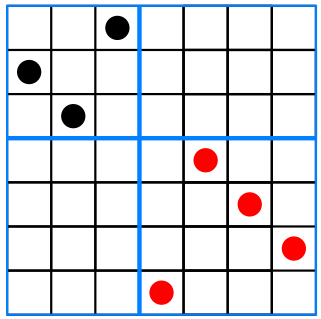


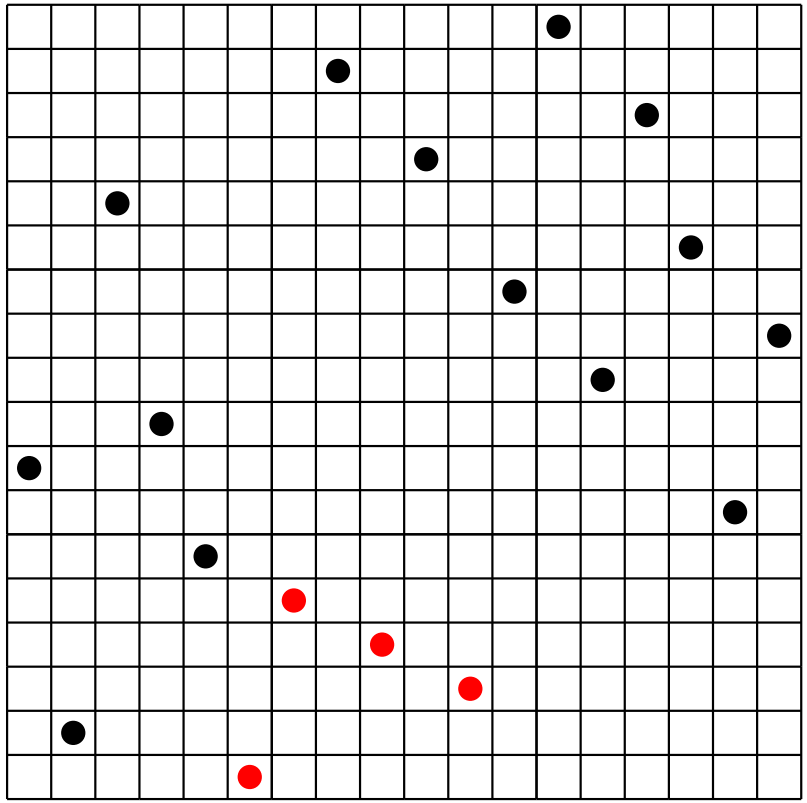
avoids





avoids





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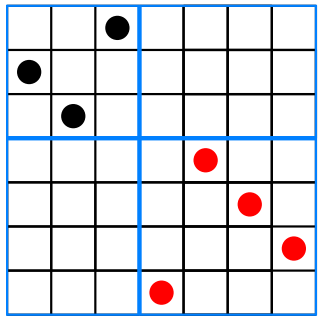

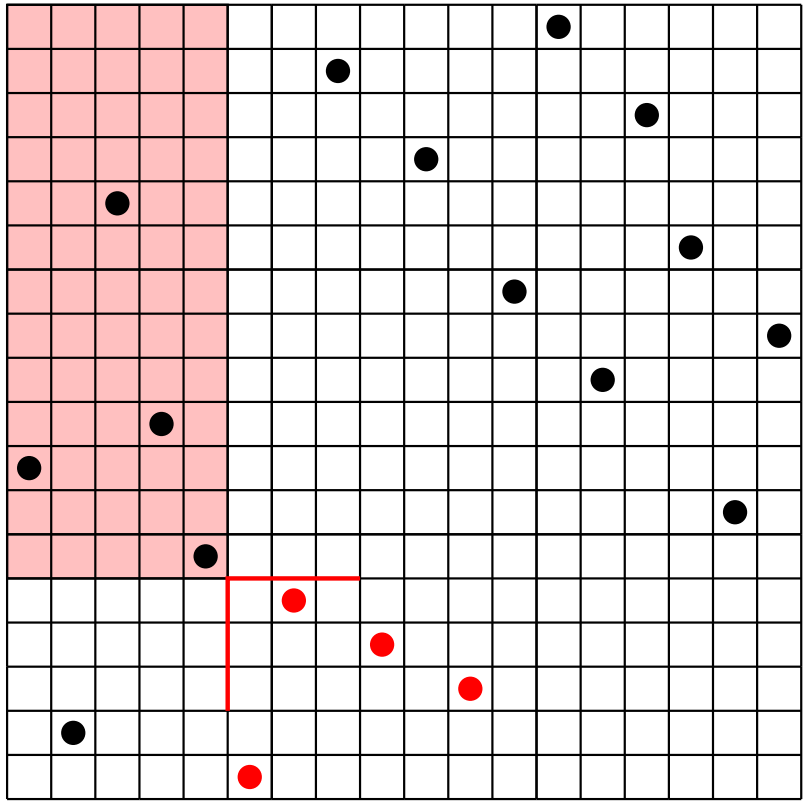

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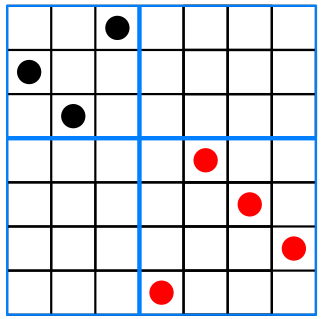

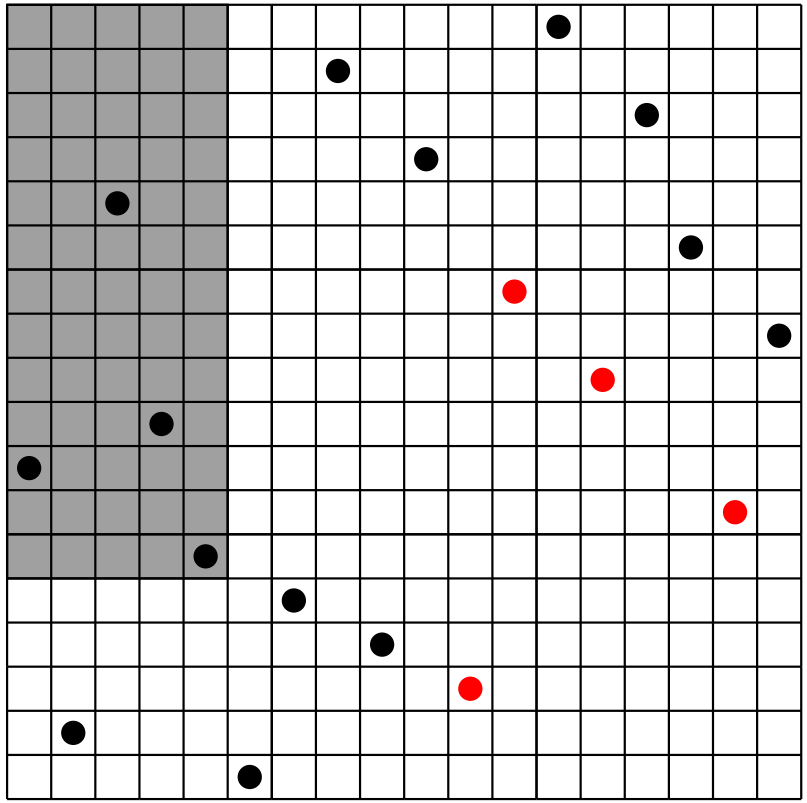

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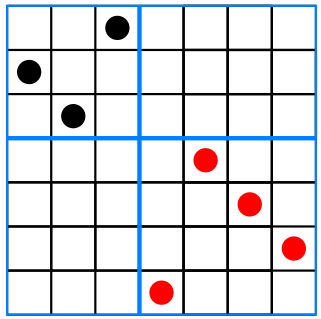

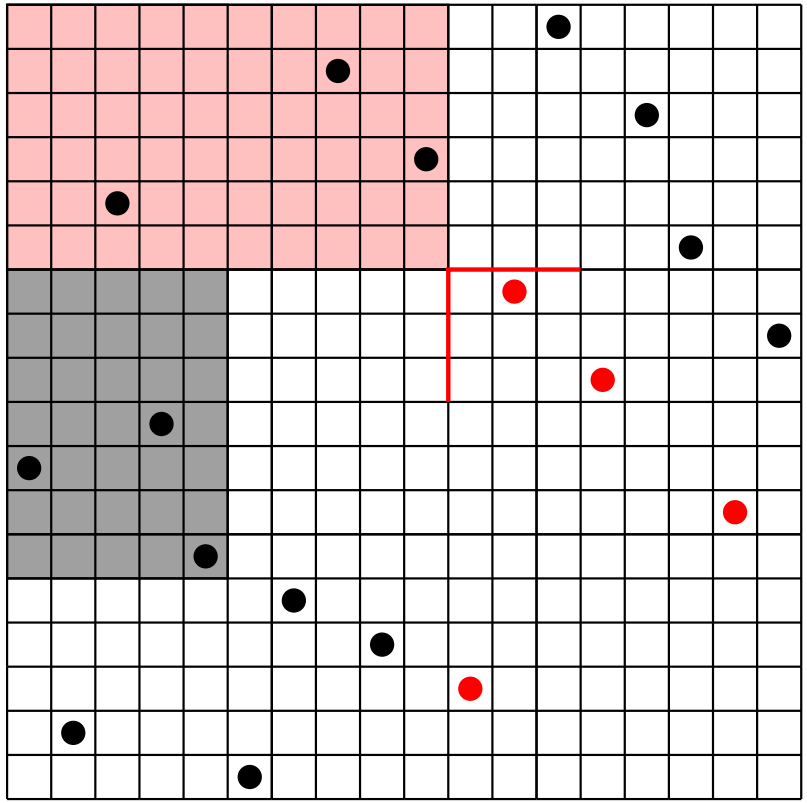

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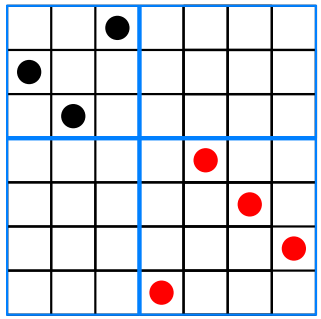

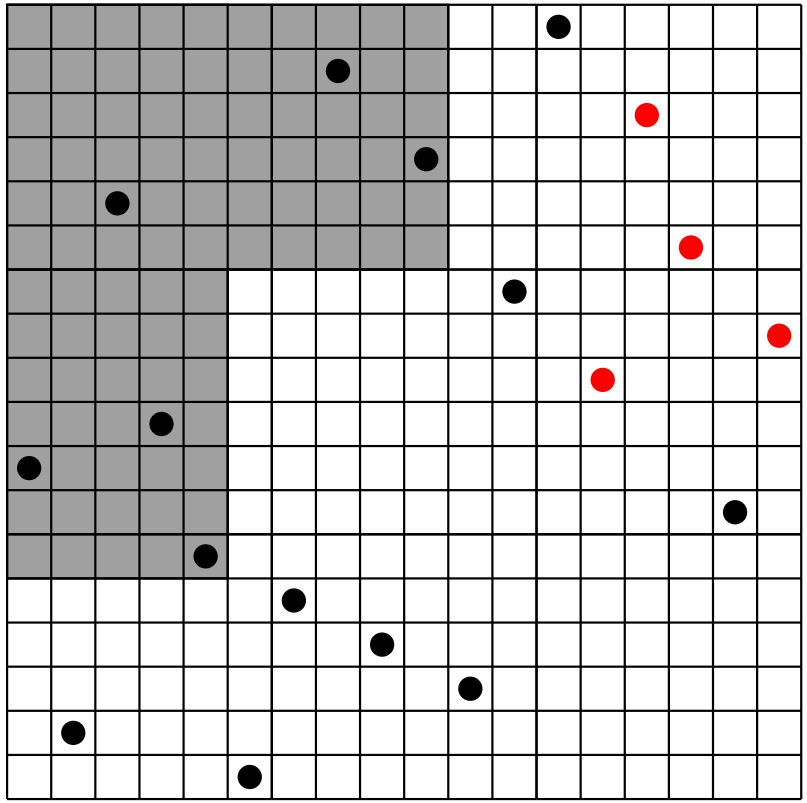

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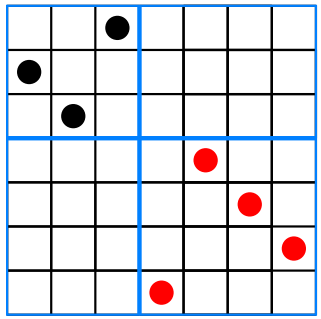

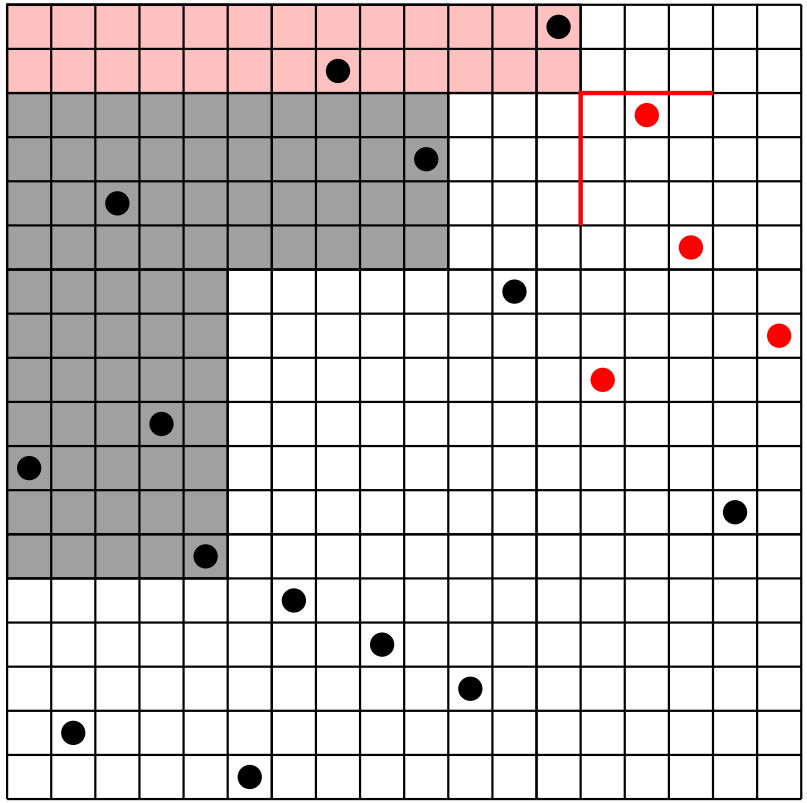

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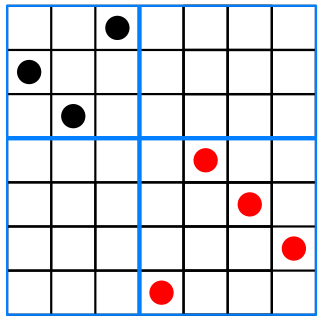
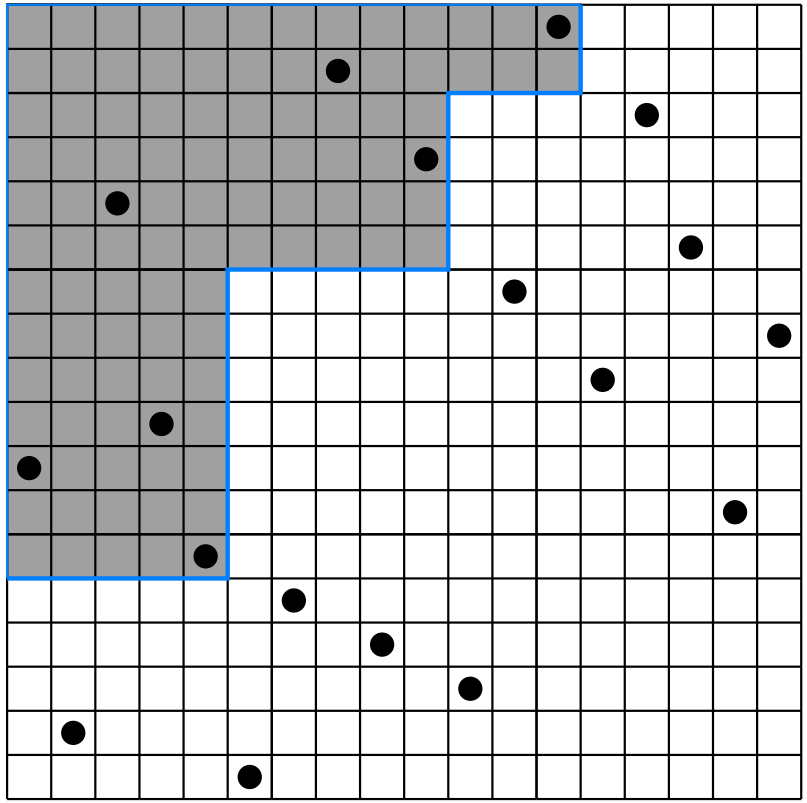
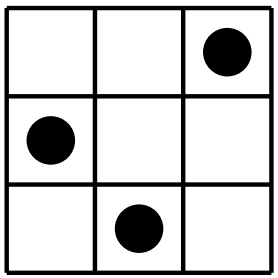
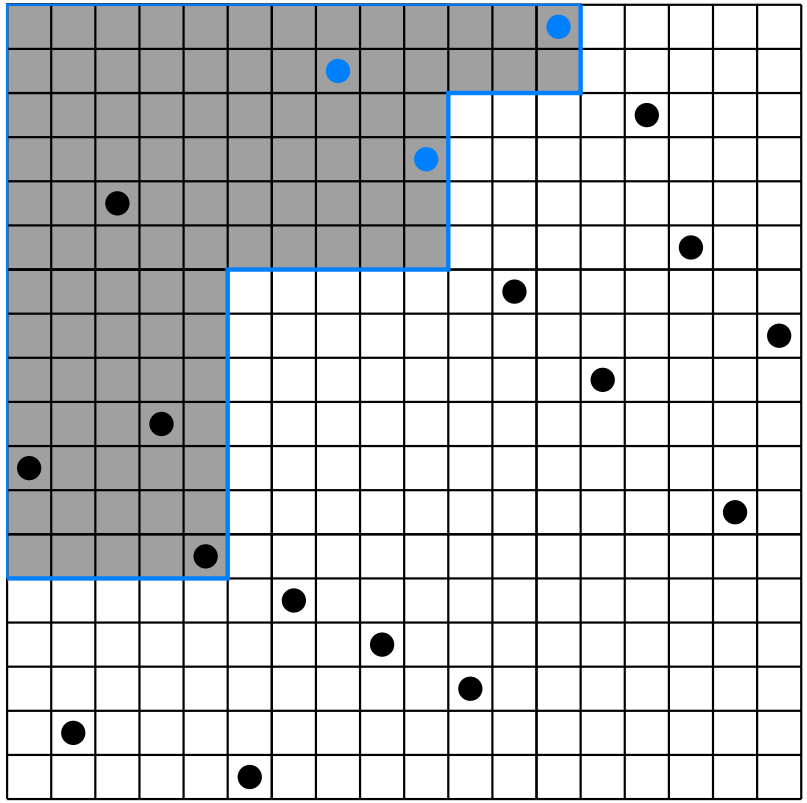


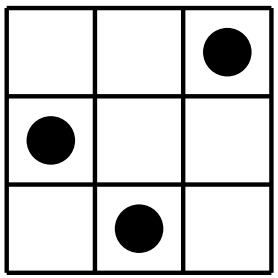


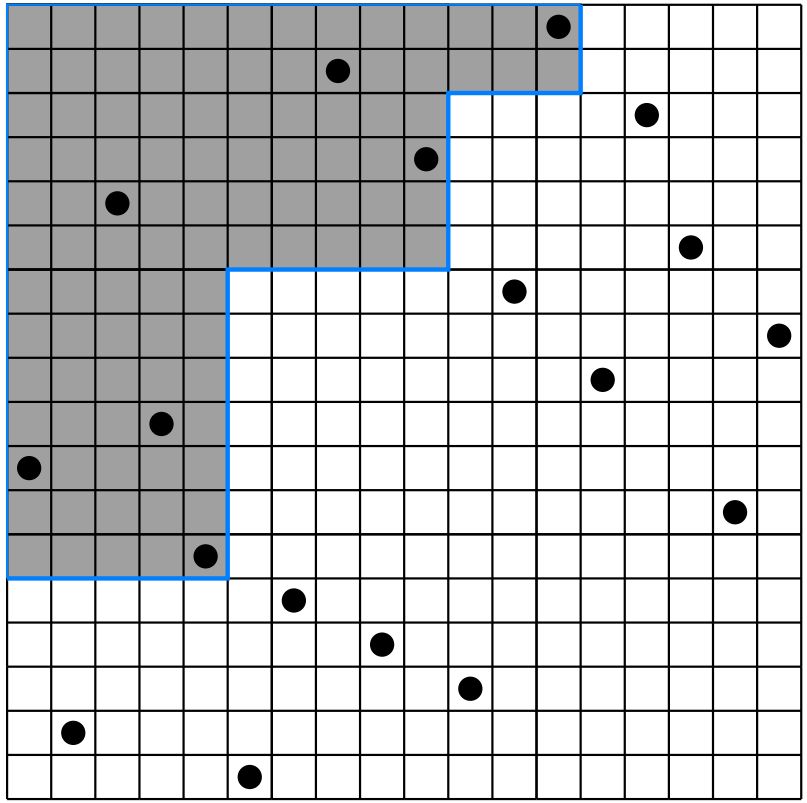
The grey region "avoids"



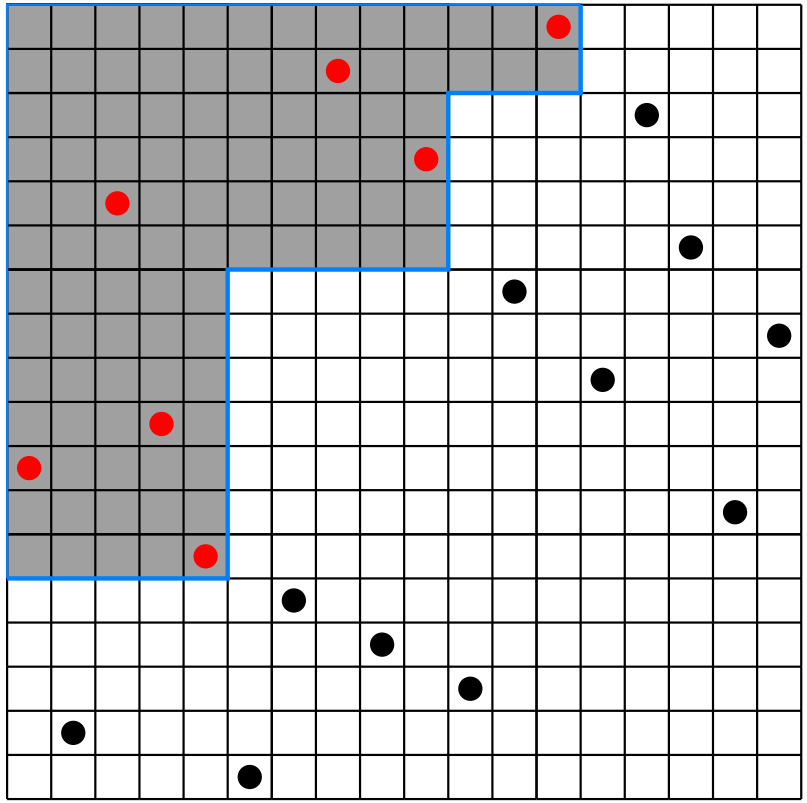


The grey region "avoids"





Cut out all rows and columns of the grey region that do not have dots.



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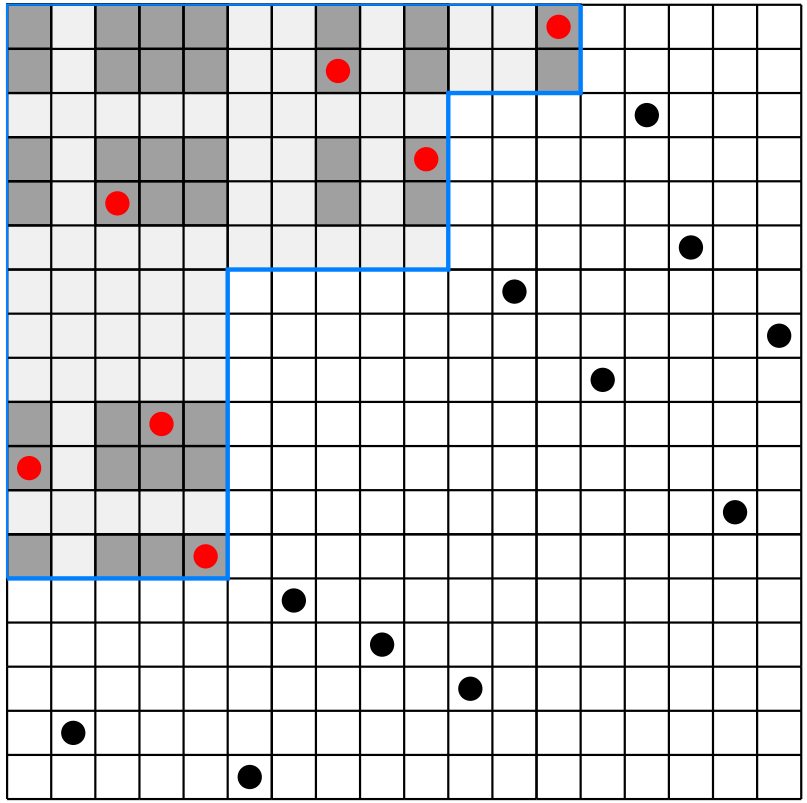

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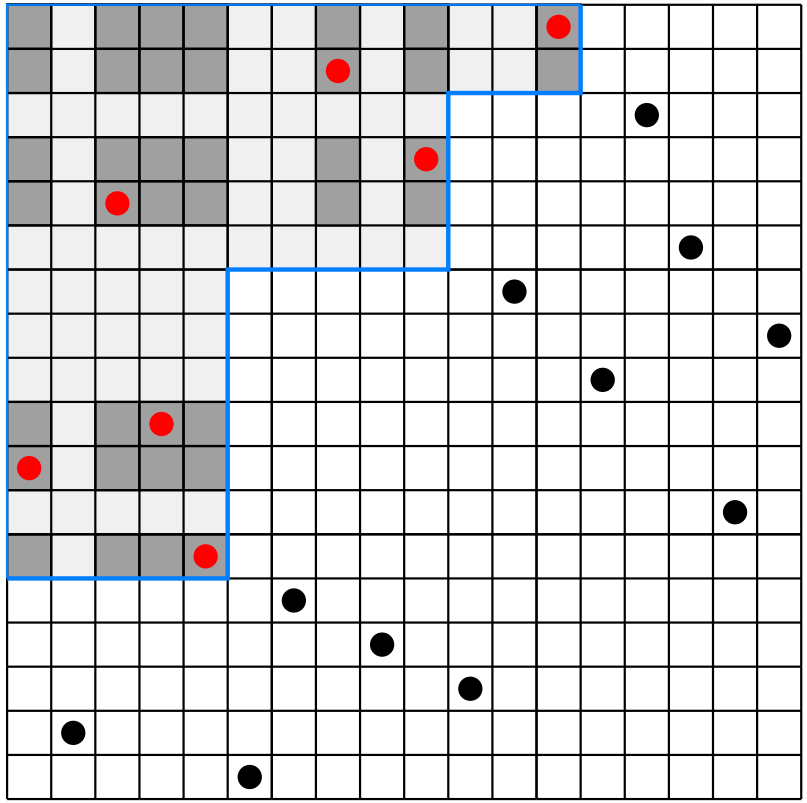

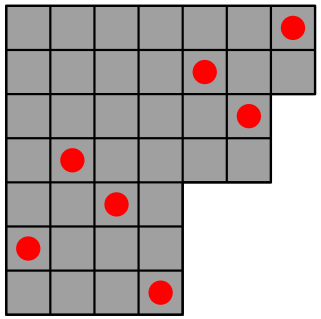

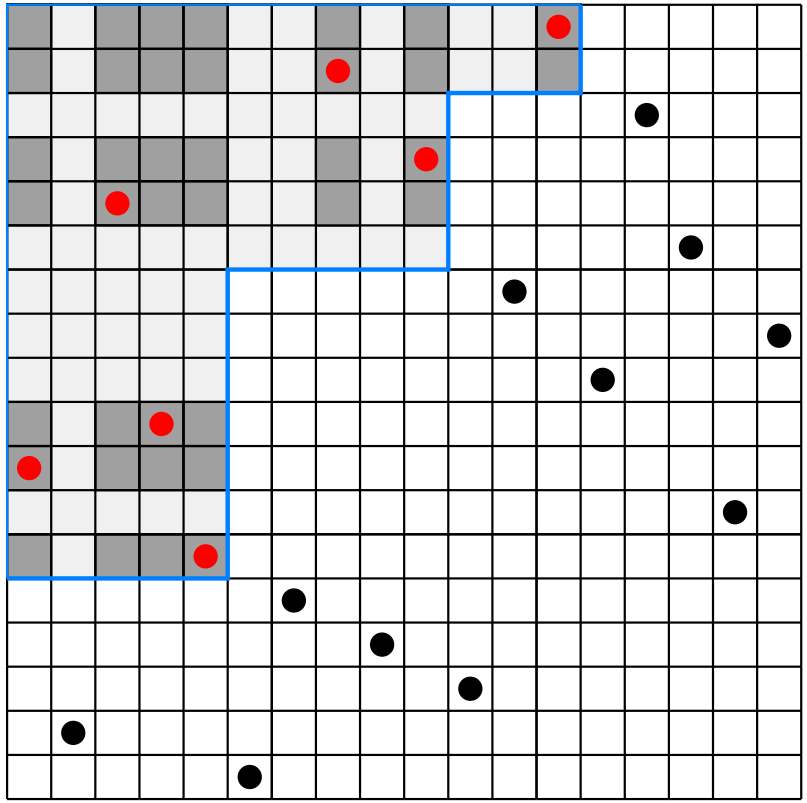

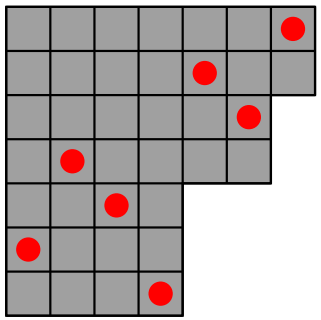

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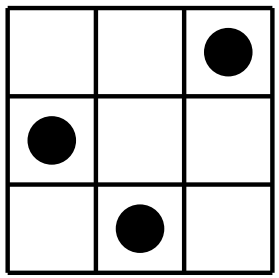

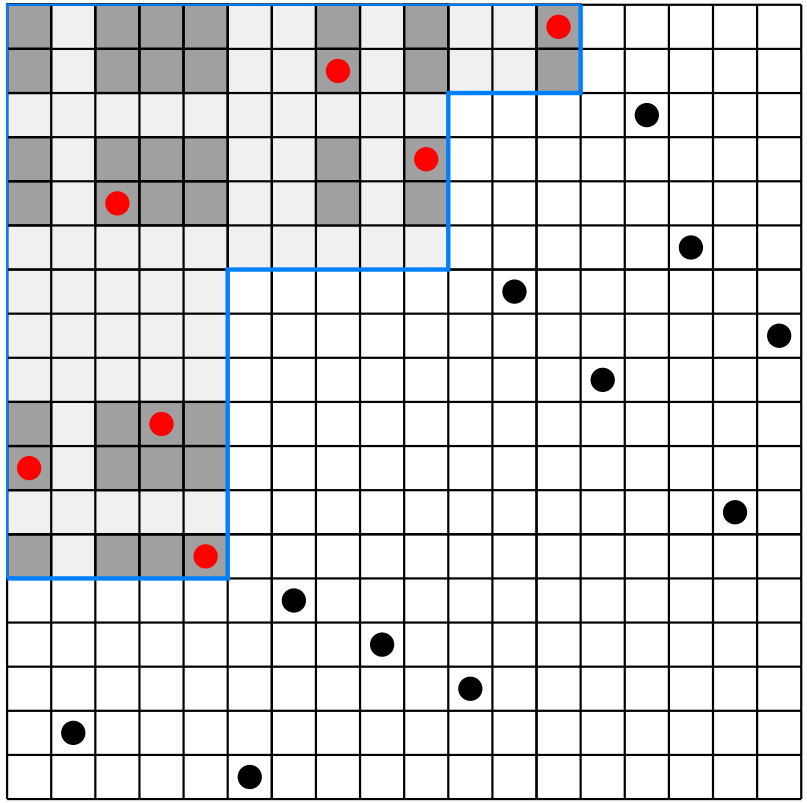

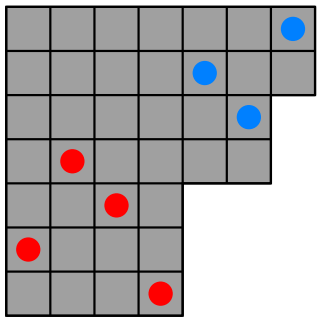

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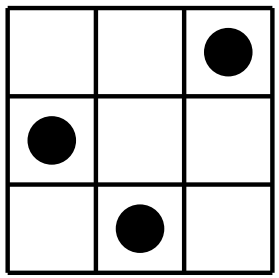
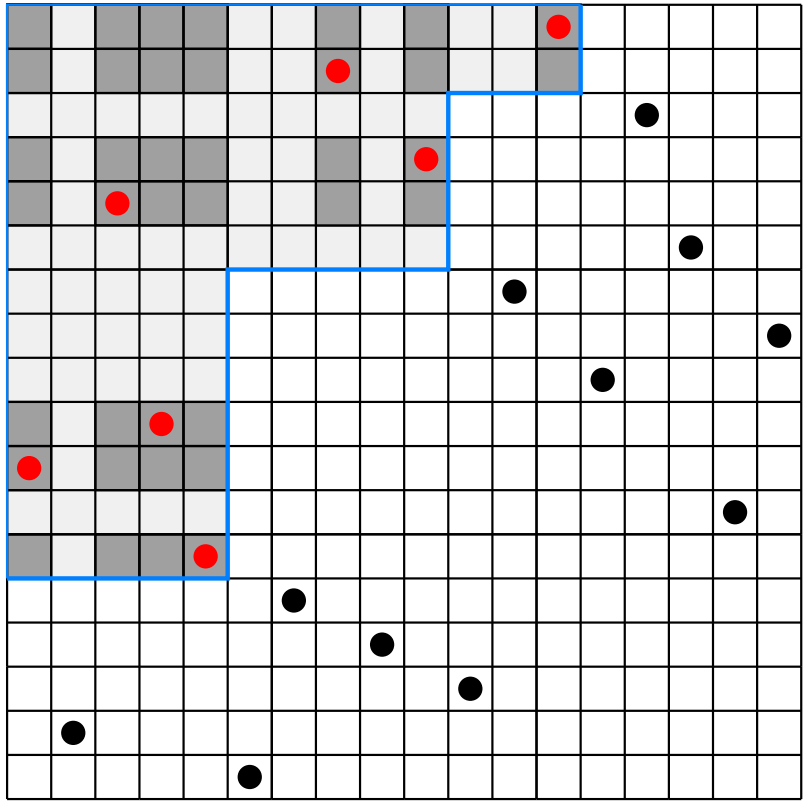
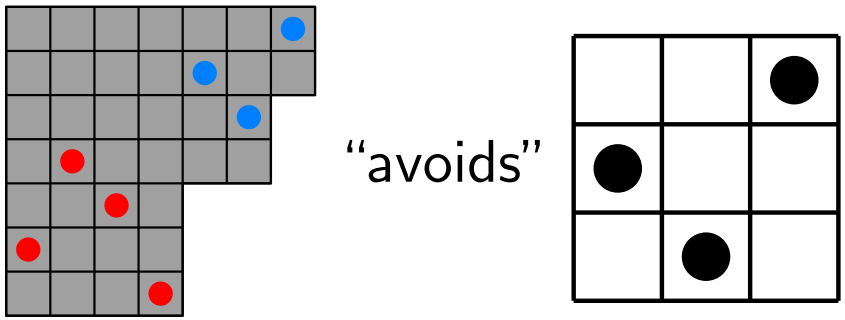
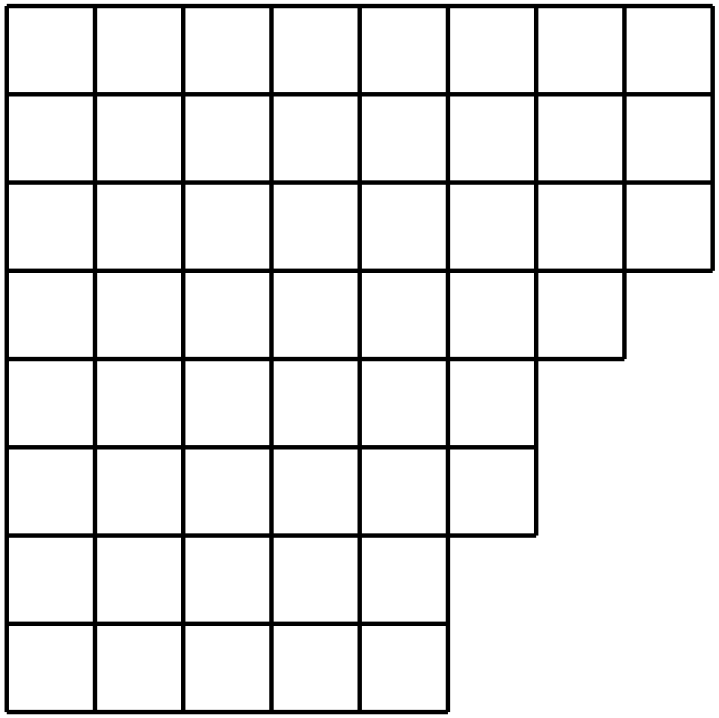


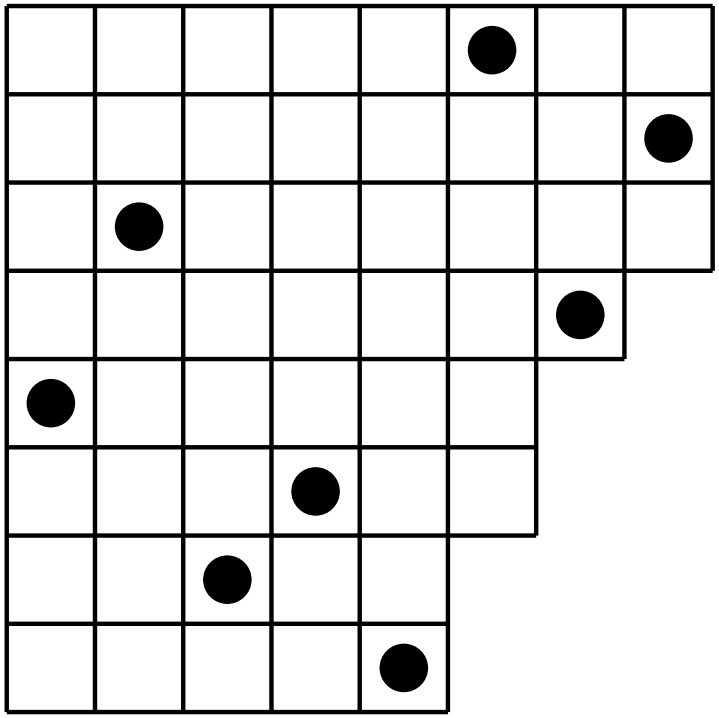




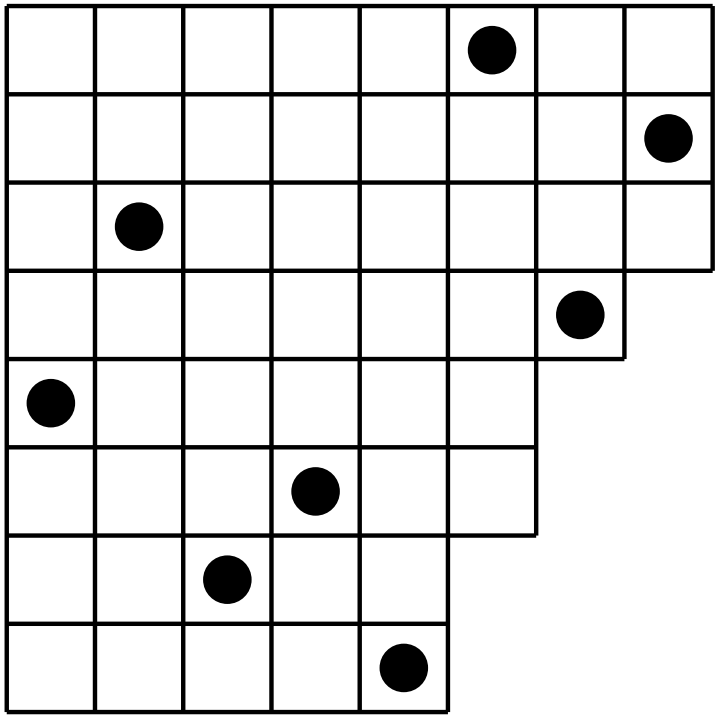
Necessary and sufficient condition.



Young diagram



Transversal in Young diagram



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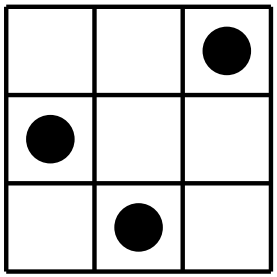
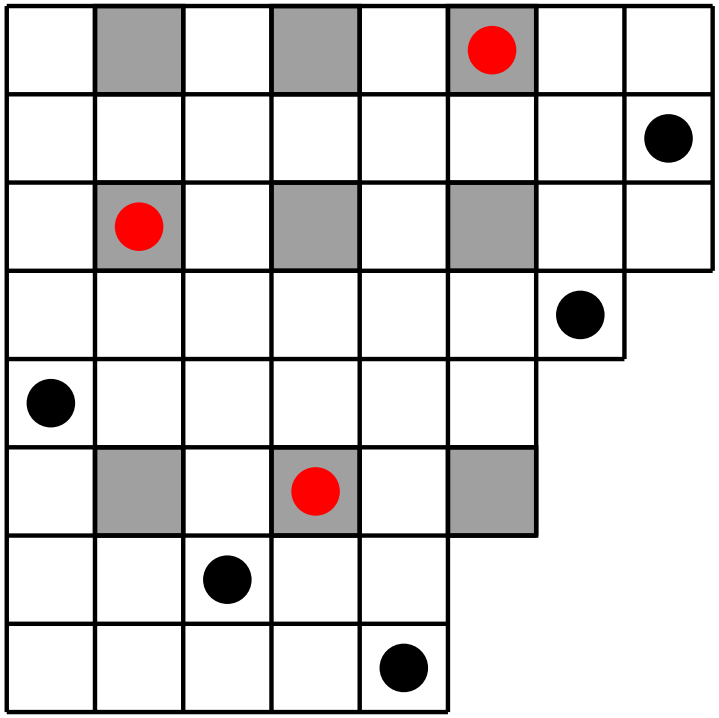


The submatrix must be completely inside the Young diagram to count as containment.



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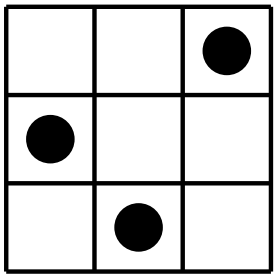

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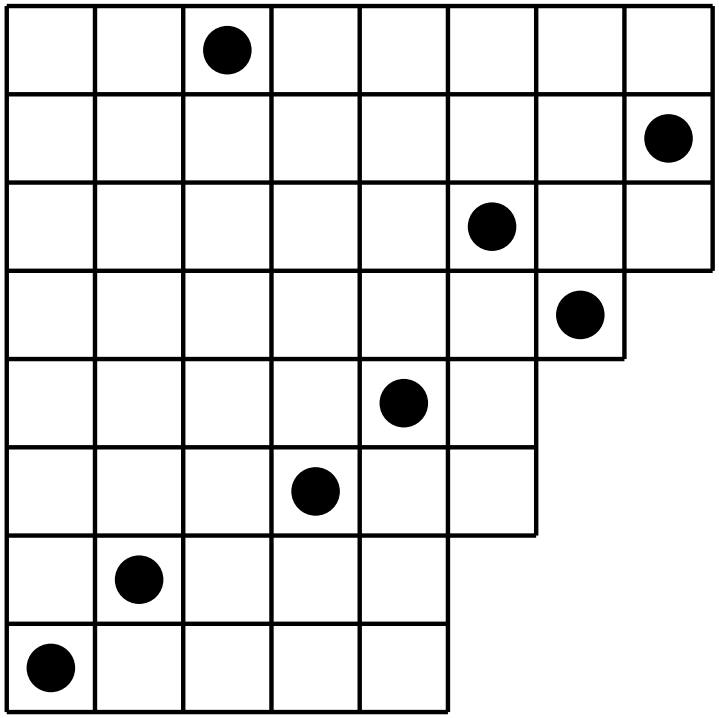
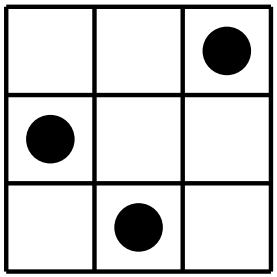
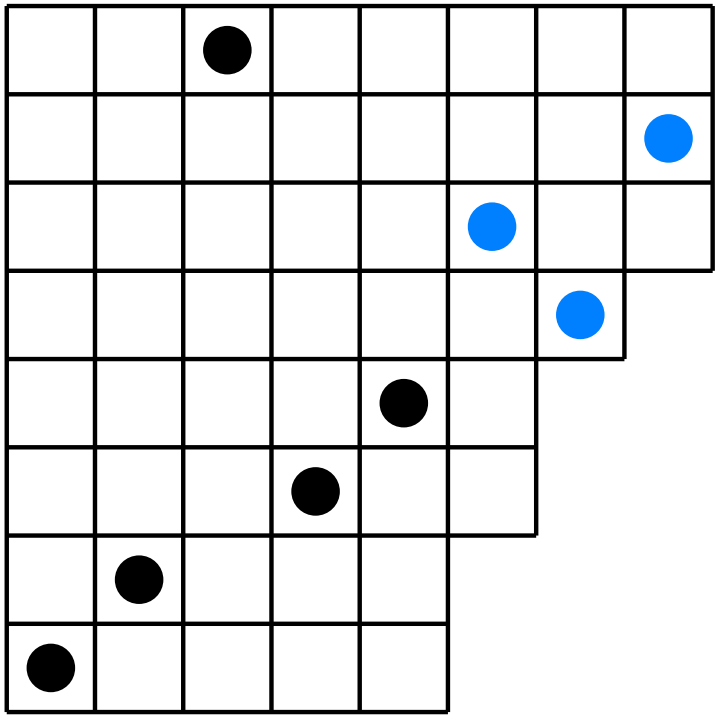


avoids





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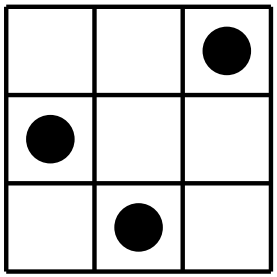

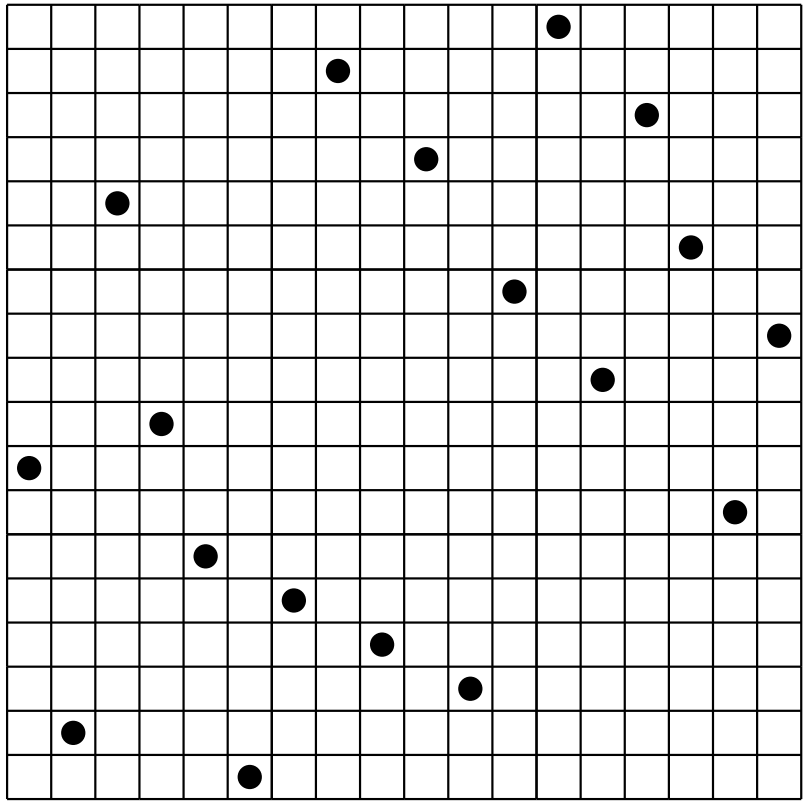

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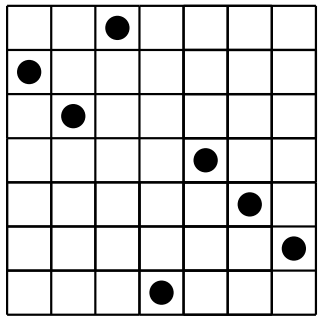

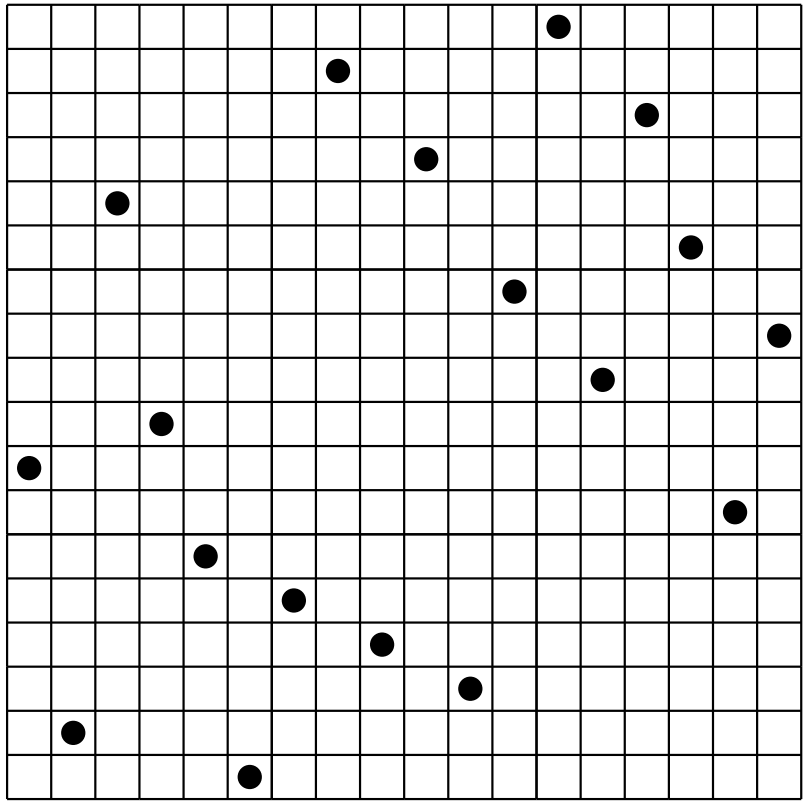

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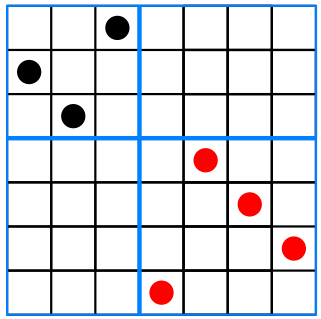

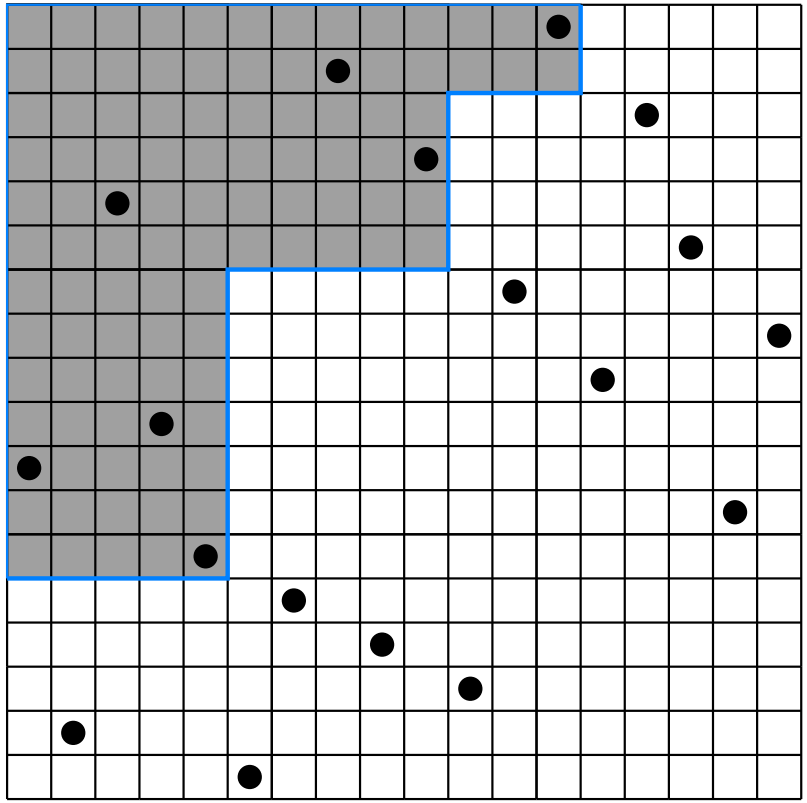

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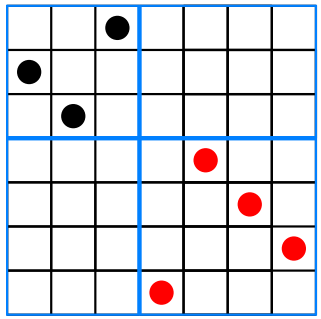
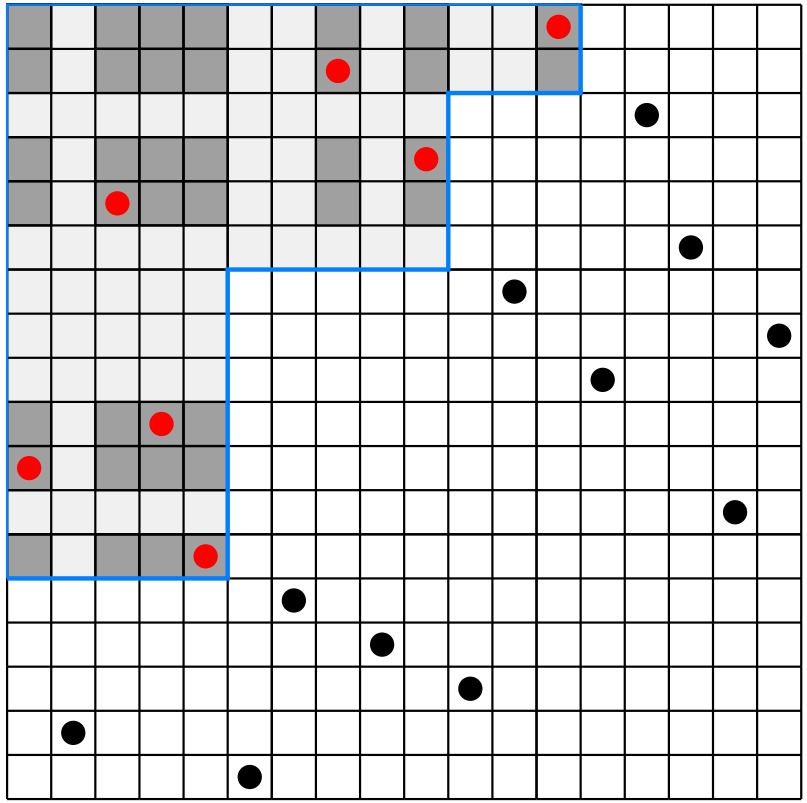
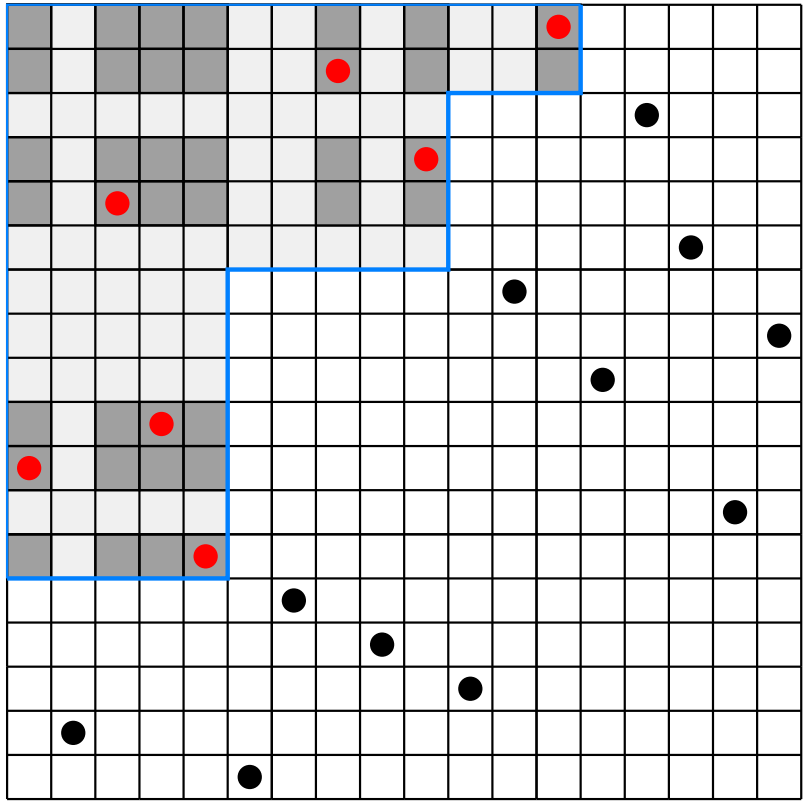
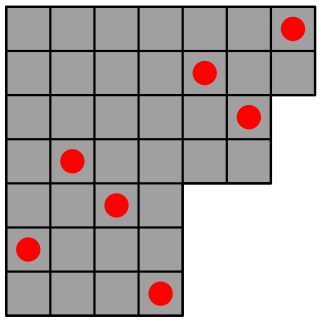
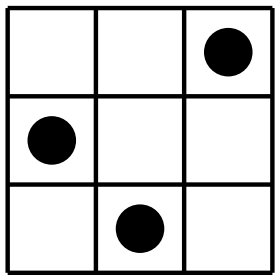








"avoids"



$S_Y(\pi)$: Set of transversals in Young diagram Y that avoid π.

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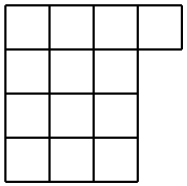
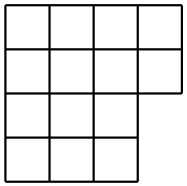
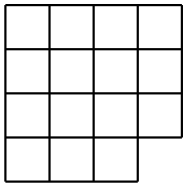
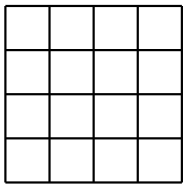
Y	$\lvert S_Y(123)\rvert$	$\lvert S_Y(321)\rvert$
	5	5
	10	10
	13	13
	14	14

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

Pattern avoidance in permutations

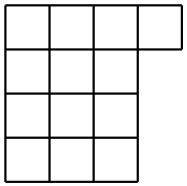
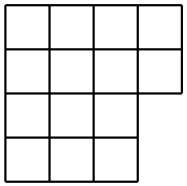
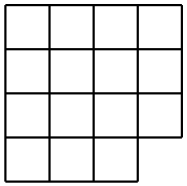
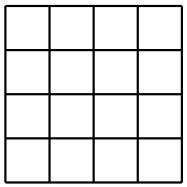
New result

Shape-Wilf-equivalence

Shape-inv-Wilf-equivalence

Even-Wilf-equivalence

$S_Y(\pi)$: Set of transversals in Young diagram Y that avoid π.

Y	$\lvert S_Y(123)\rvert$	$\lvert S_Y(321)\rvert$
	5	5
	10	10
	13	13
	14	14

In fact, $\lvert S_Y(123)\rvert = \lvert S_Y(321)\rvert$ for all Young diagrams Y.

An infinite family of inv-Wilf-equivalent permutation pairs

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Pattern avoidance in permutations

New result

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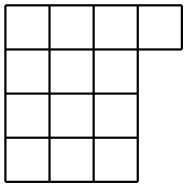
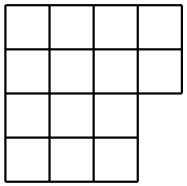
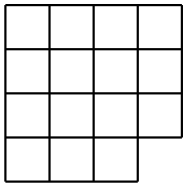
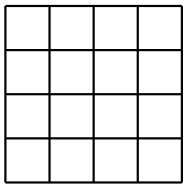
Y	$\lvert S_Y(123)\rvert$	$\lvert S_Y(321)\rvert$
	5	5
	10	10
	13	13
	14	14

In fact, $\lvert S_Y(123)\rvert = \lvert S_Y(321)\rvert$ for all Young diagrams Y. (123) and (321) are **shape-Wilf-equivalent**.

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

Pattern avoidance in permutations

New result

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Even-Wilf-equivalence

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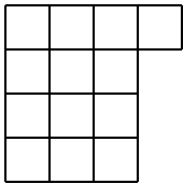
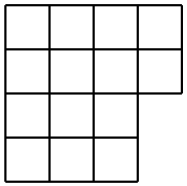
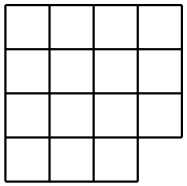
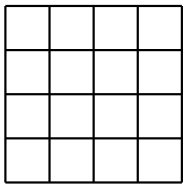
Y	$\lvert S_Y(123)\rvert$	$\lvert S_Y(321)\rvert$
	5	5
	10	10
	13	13
	14	14

In fact, $|S_Y(123)| = |S_Y(321)|$ for all Young diagrams Y.
(123) and (321) are **shape-Wilf-equivalent**.
Shape-Wilf-equivalence implies Wilf-equivalence.

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

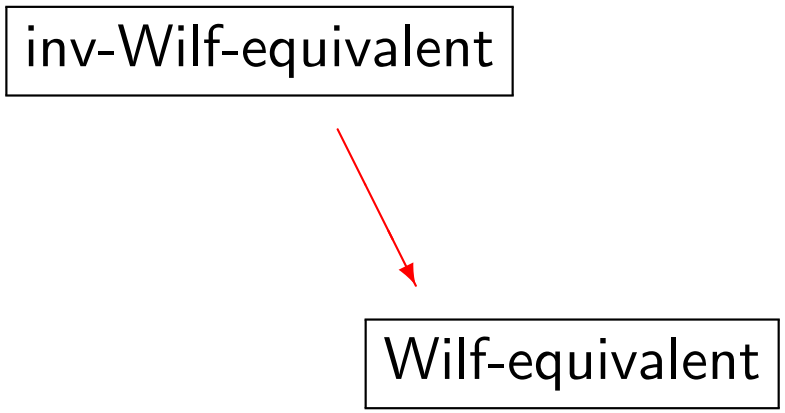
Pattern avoidance in permutations

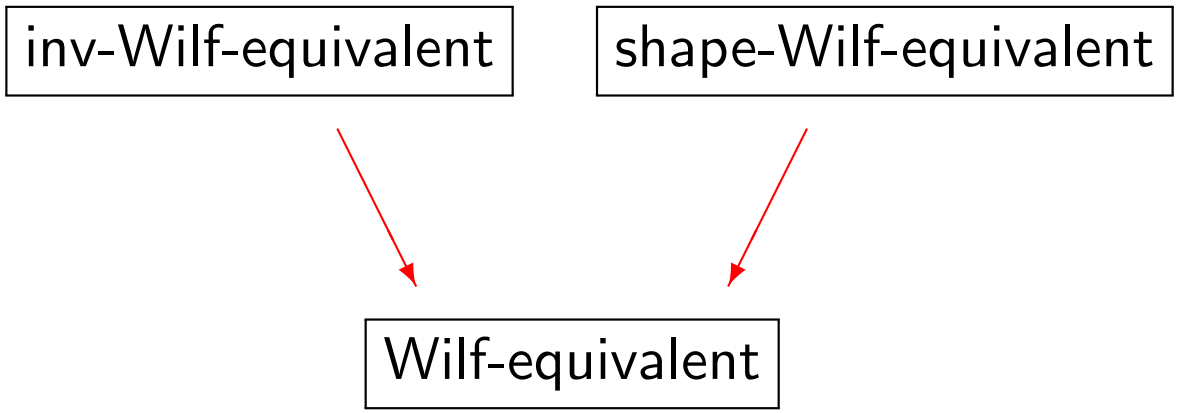
New result

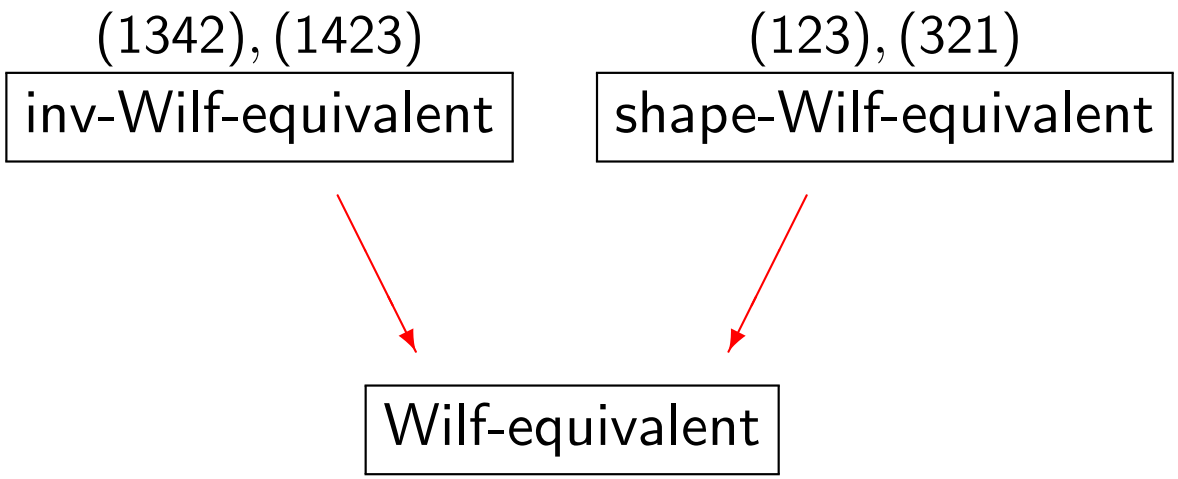
Shape-Wilf-equivalence

Shape-inv-Wilf-equivalence

Even-Wilf-equivalence


inv-Wilf-equivalent
Wilf-equivalent


inv-Wilf-equivalent
shape-Wilf-equivalent
Wilf-equivalent


(1342), (1423)
inv-Wilf-equivalent
(123), (321)
shape-Wilf-equivalent
Wilf-equivalent

- Stankova, West (2002): (231) and (312) are shape-Wilf-equivalent.

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- Backelin, West, Xin (2007): If α and β are shape-Wilf-equivalent, then so are $\alpha \oplus \gamma$ and $\beta \oplus \gamma$ for all permutations γ.

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So $(231) \oplus \gamma$ and $(312) \oplus \gamma$ are shape-Wilf-equivalent for all permutations γ.

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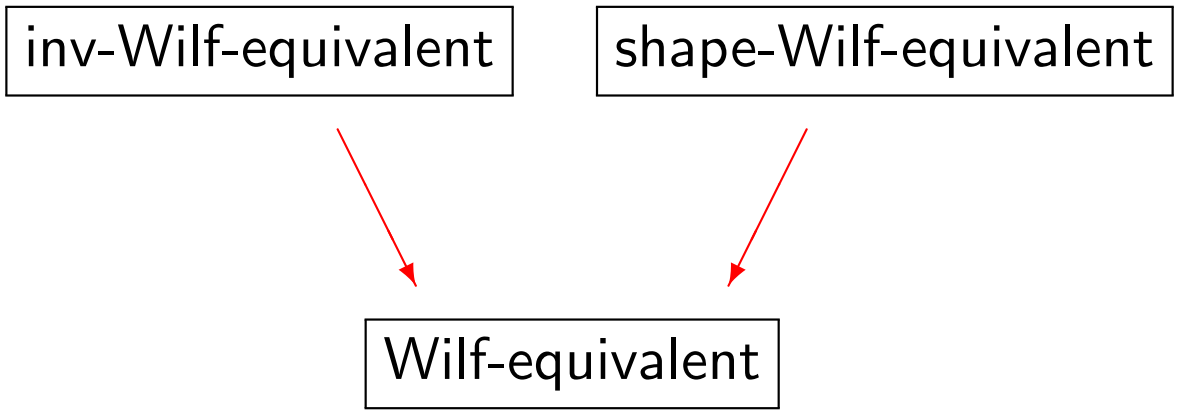
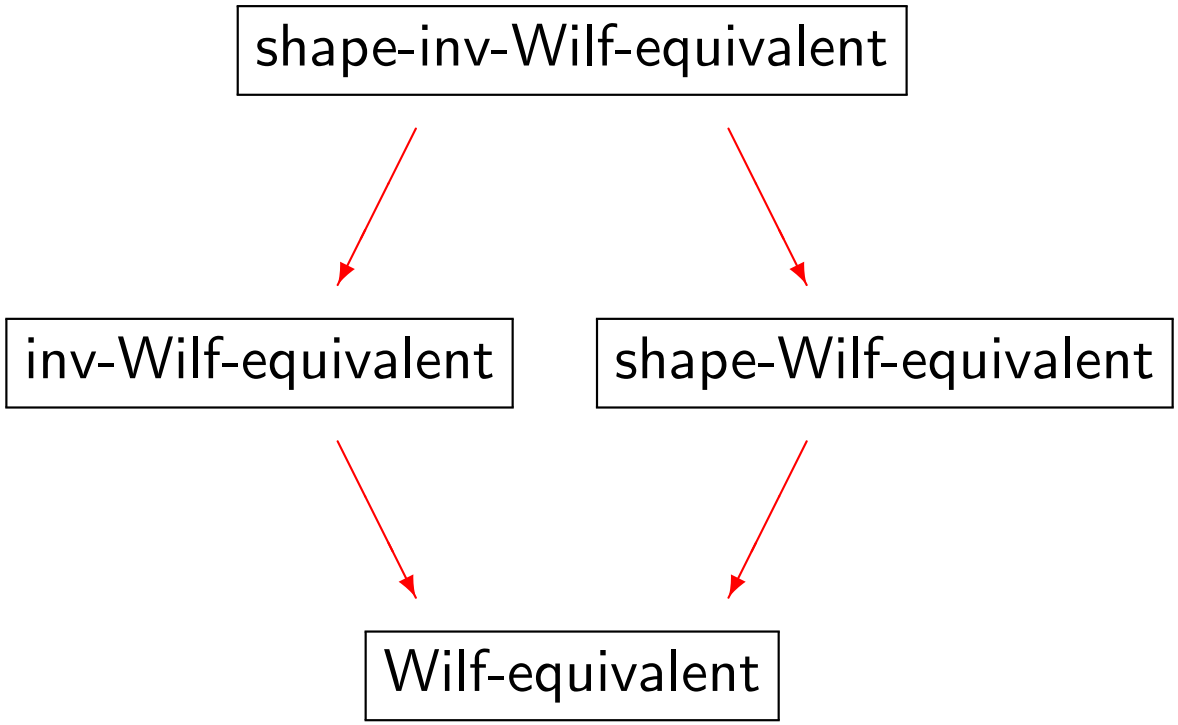
So $(231) \oplus \gamma$ and $(312) \oplus \gamma$ are shape-Wilf-equivalent for all permutations γ.
How do we show that they are inv-Wilf-equivalent?


inv-Wilf-equivalent
shape-Wilf-equivalent
Wilf-equivalent


shape-inv-Wilf-equivalent
inv-Wilf-equivalent
shape-Wilf-equivalent
Wilf-equivalent

Combine shape-Wilf-equivalence and inv-Wilf-equivalence to define **shape-inv-Wilf-equivalence**: For each Young diagram Y, the sets of transversals in Y avoiding each of the two permutations have the same distribution according to inversion number.

Combine shape-Wilf-equivalence and inv-Wilf-equivalence to define **shape-inv-Wilf-equivalence**: For each Young diagram Y, the sets of transversals in Y avoiding each of the two permutations have the same distribution according to inversion number.
To show that $(231) \oplus \gamma$ and $(312) \oplus \gamma$ are inv-Wilf-equivalent for all permutations γ, we just need the following.

- Prove that if α and β are shape-inv-Wilf-equivalent, then so are $\alpha \oplus \gamma$ and $\beta \oplus \gamma$.

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- Prove that (231) and (312) are shape-inv-Wilf-equivalent.

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  - Extend Backelin-West-Xin result on shape-Wilf-equivalence of $\alpha \oplus \gamma$ and $\beta \oplus \gamma$.

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- Prove that (231) and (312) are shape-inv-Wilf-equivalent.
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- Prove that (231) and (312) are shape-inv-Wilf-equivalent.
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  - $I(Y)$: inv-polynomial (generating function) for transversals in Young diagram Y that avoid (231).

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  - The "reflection" of row decomposition called **column decomposition** (prove by induction).

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

Pattern avoidance in permutations

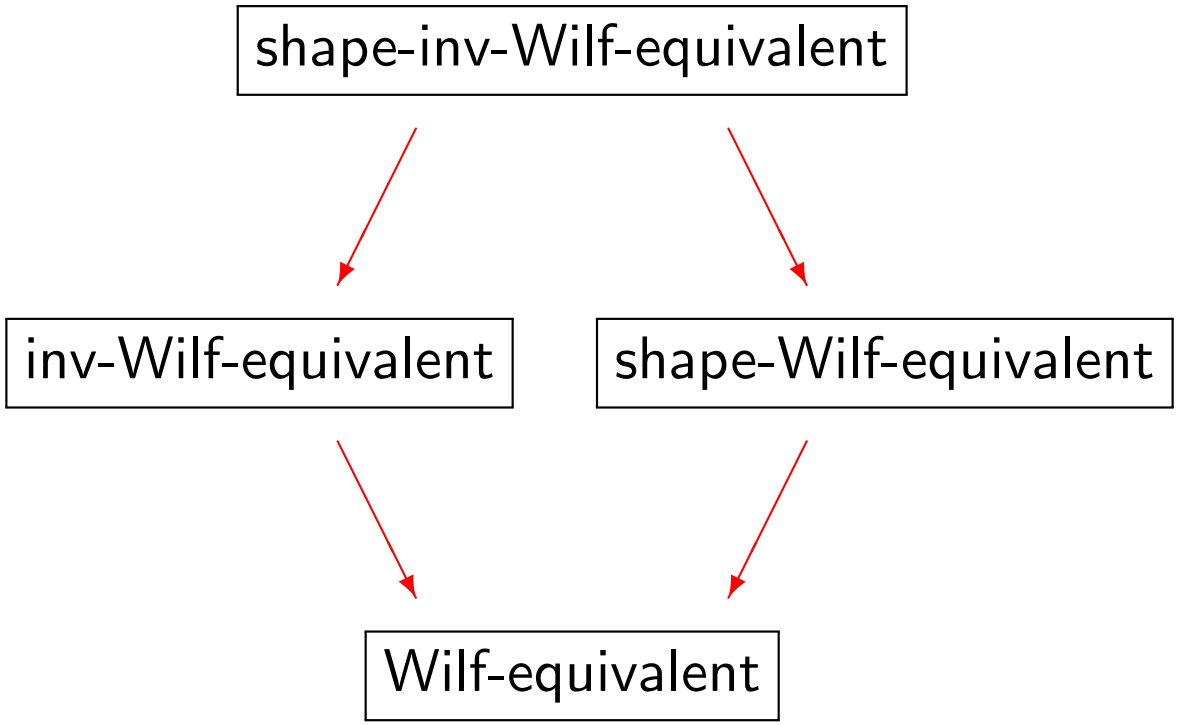
New result

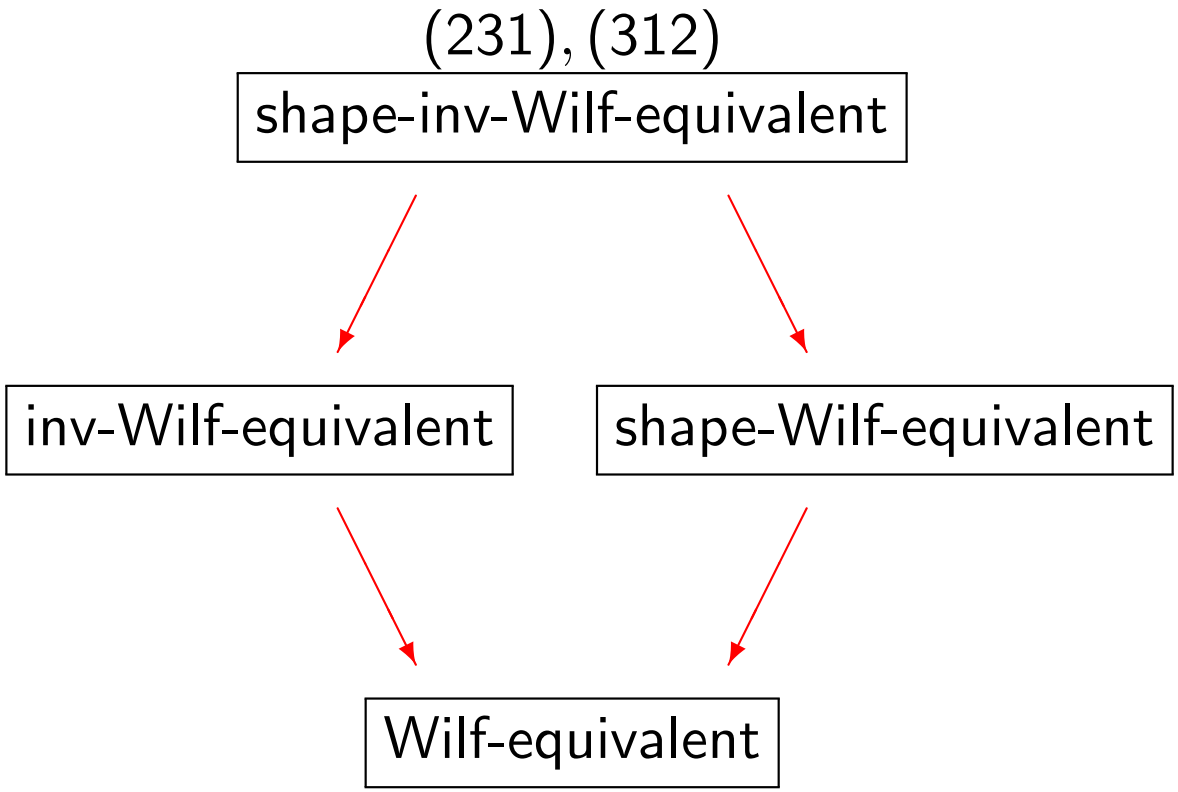
Shape-Wilf-equivalence

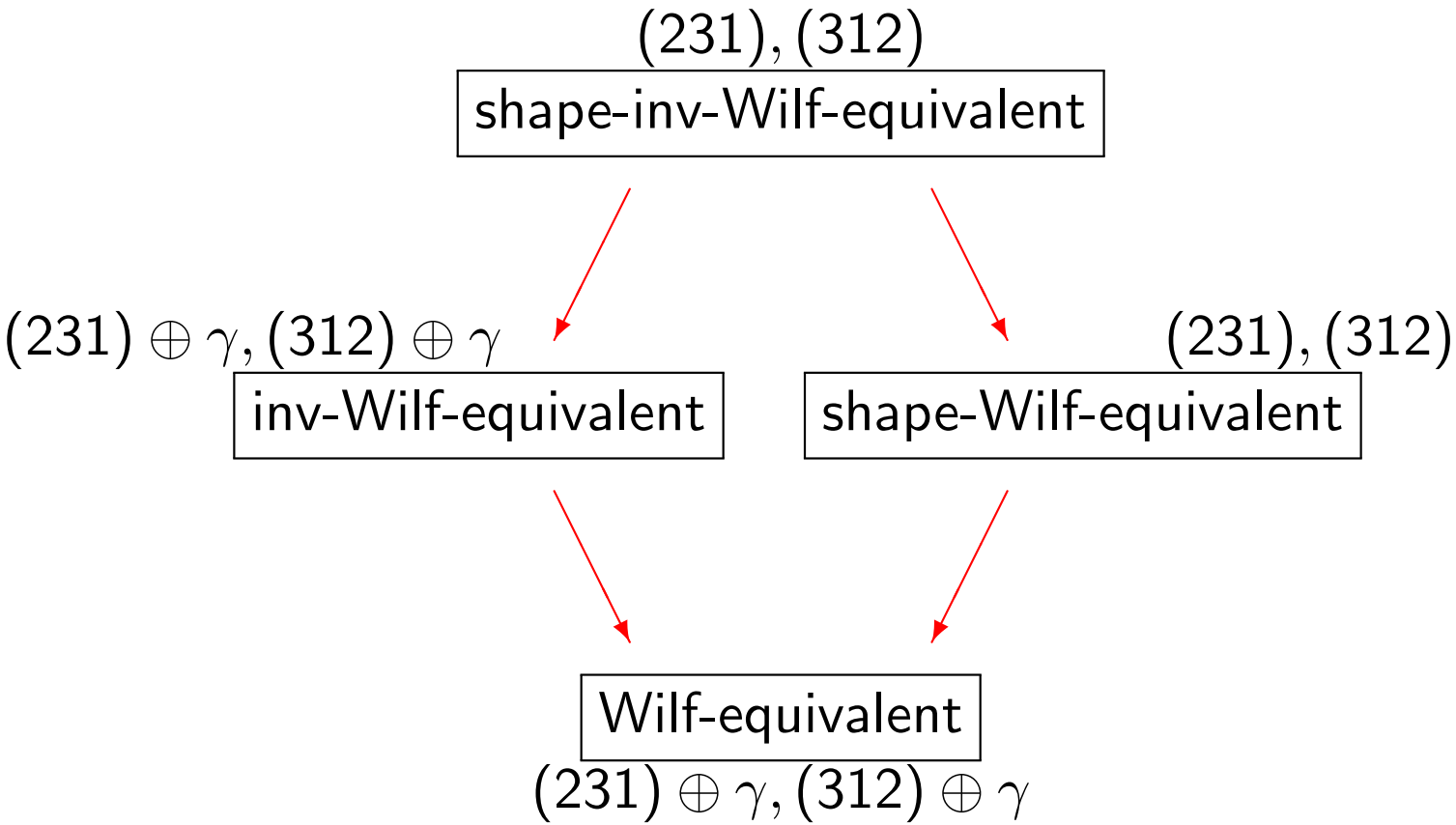
Shape-inv-Wilf-equivalence

Even-Wilf-equivalence

- Prove that if α and β are shape-inv-Wilf-equivalent, then so are $\alpha \oplus \gamma$ and $\beta \oplus \gamma$.
  - Extend Backelin-West-Xin result on shape-Wilf-equivalence of $\alpha \oplus \gamma$ and $\beta \oplus \gamma$.
- Prove that (231) and (312) are shape-inv-Wilf-equivalent.
  - Extend Stankova-West result on shape-Wilf-equivalence of (231) and (312).
  - $I(Y)$: inv-polynomial (generating function) for transversals in Young diagram Y that avoid (231).
  - A functional relation called **row decomposition** (expressing $I(Y)$ in terms of "smaller" $I(Z)$).
  - The "reflection" of row decomposition called **column decomposition** (prove by induction).
  - Together, these show that (231) and (312) are shape-inv-Wilf-equivalent.


shape-inv-Wilf-equivalent
inv-Wilf-equivalent
shape-Wilf-equivalent
Wilf-equivalent


(231), (312)
shape-inv-Wilf-equivalent
inv-Wilf-equivalent
shape-Wilf-equivalent
Wilf-equivalent


(231), (312)
shape-inv-Wilf-equivalent
(231) ⊕ γ, (312) ⊕ γ
inv-Wilf-equivalent
(231), (312)
shape-Wilf-equivalent
Wilf-equivalent
(231) ⊕ γ, (312) ⊕ γ

We now define even-Wilf-equivalence, introduced by Baxter and Jaggard in 2011.

$S_4(231)$	inv	$S_4(312)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (2143) (2314)
(1432) (3214) (4123)	3	(1432) (3214) (2341)
(4132) (4213)	4	(2431) (3241)
(4312)	5	(3421)
(4321)	6	(4321)

$S_4(231)$	inv	$S_4(312)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (2143) (2314)
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(4312)	5	(3421)
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$S_4(231)$	inv	$S_4(312)$
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(1243) (1324) (2134)	1	(1243) (1324) (2134)
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(4132) (4213)	4	(2431) (3241)
(4312)	5	(3421)
(4321)	6	(4321)

Red denotes even permutation. (A permutation is even if and only if its inversion number is even.)

$S_4(231)$	inv	$S_4(312)$
(1234)	0	(1234)
(1243) (1324) (2134)	1	(1243) (1324) (2134)
(1423) (2143) (3124)	2	(1342) (2143) (2314)
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$A_n(\pi)$: Set of even permutations of length n that avoid π.

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(4321)	6	(4321)

$A_n(\pi)$: Set of even permutations of length n that avoid π.
$|A_4(231)| = |A_4(312)| = 7$, and similarly
$|A_n(231)| = |A_n(312)|$ for all n.

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(1234)	0	(1234)
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(1423) (2143) (3124)	2	(1342) (2143) (2314)
(1432) (3214) (4123)	3	(1432) (3214) (2341)
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(4321)	6	(4321)

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(231) and (312) are **even-Wilf-equivalent**.

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

Pattern avoidance in permutations

New result

Shape-Wilf-equivalence

Shape-inv-Wilf-equivalence

Even-Wilf-equivalence

$S_4(231)$	inv	$S_4(312)$
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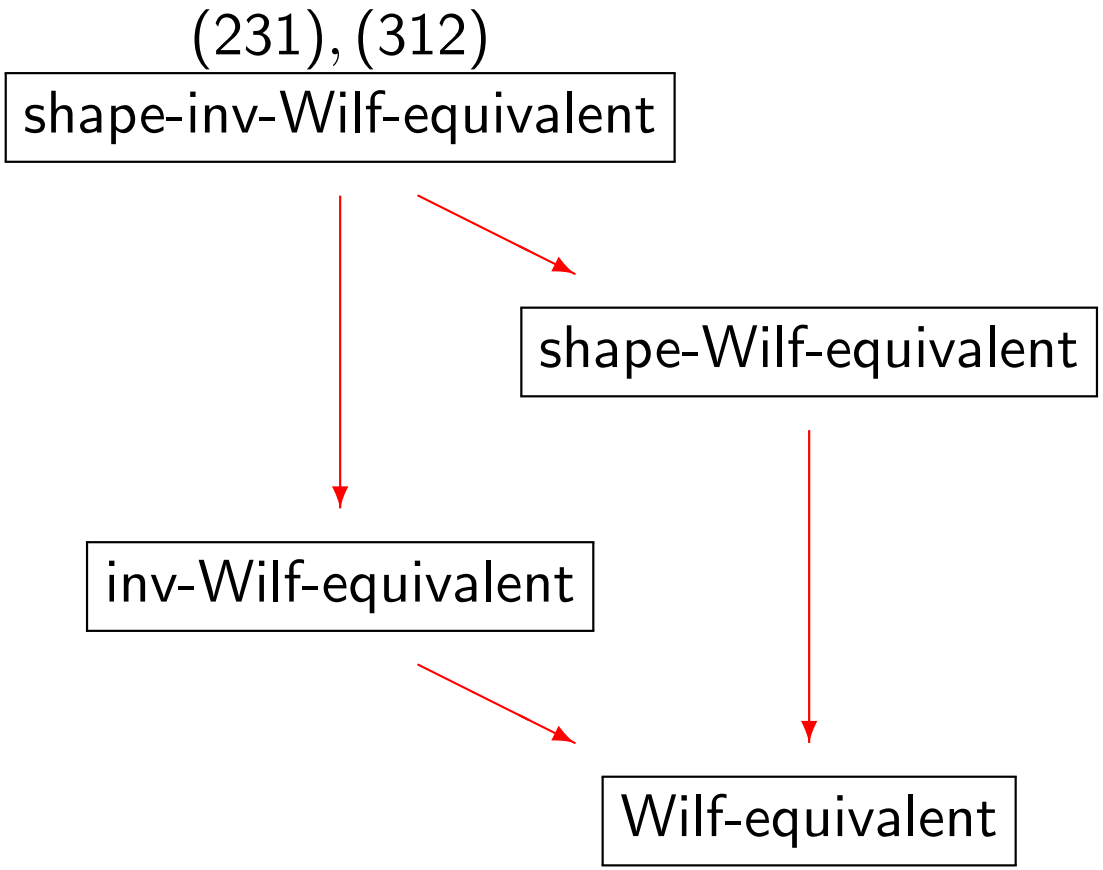
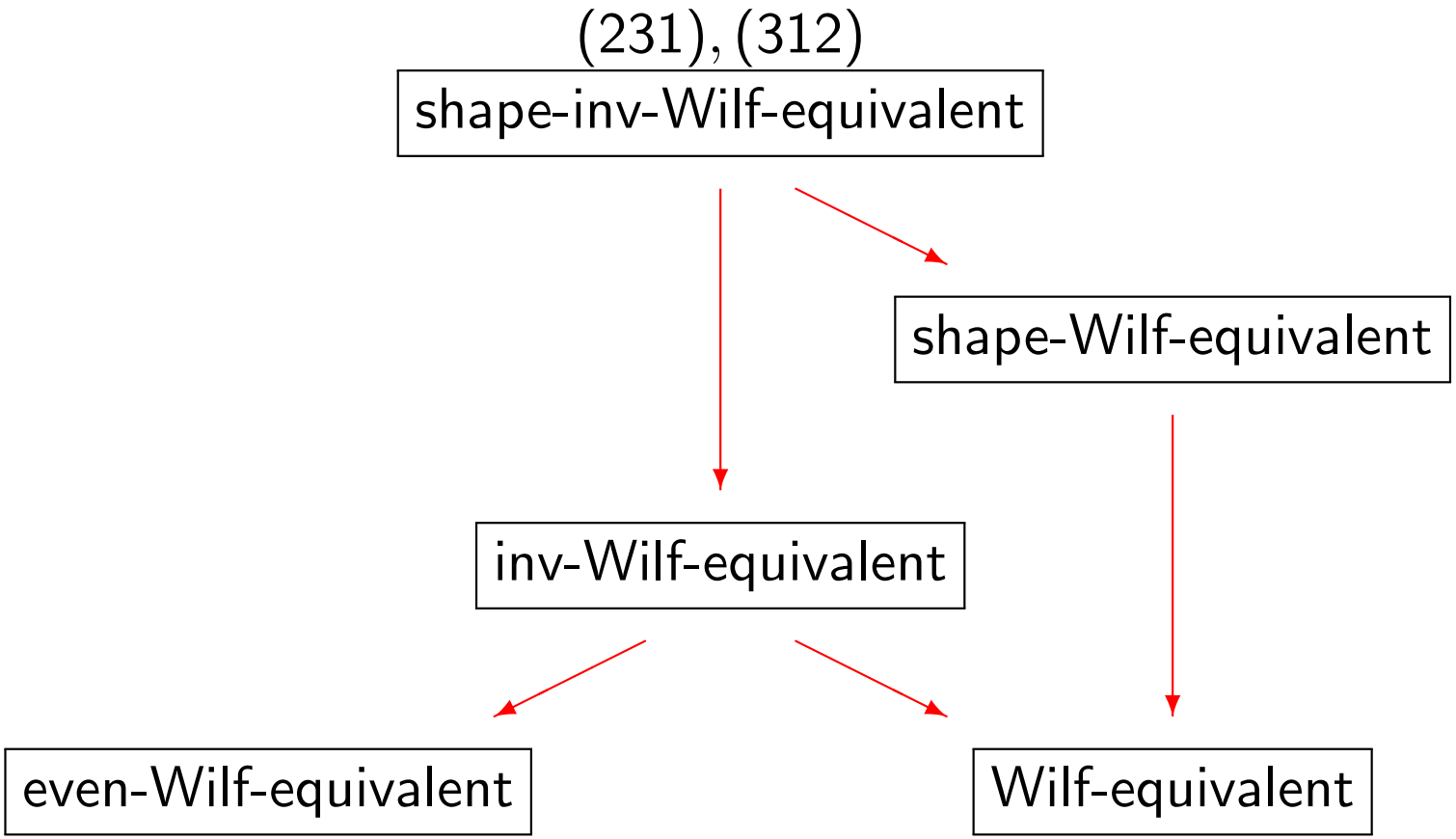
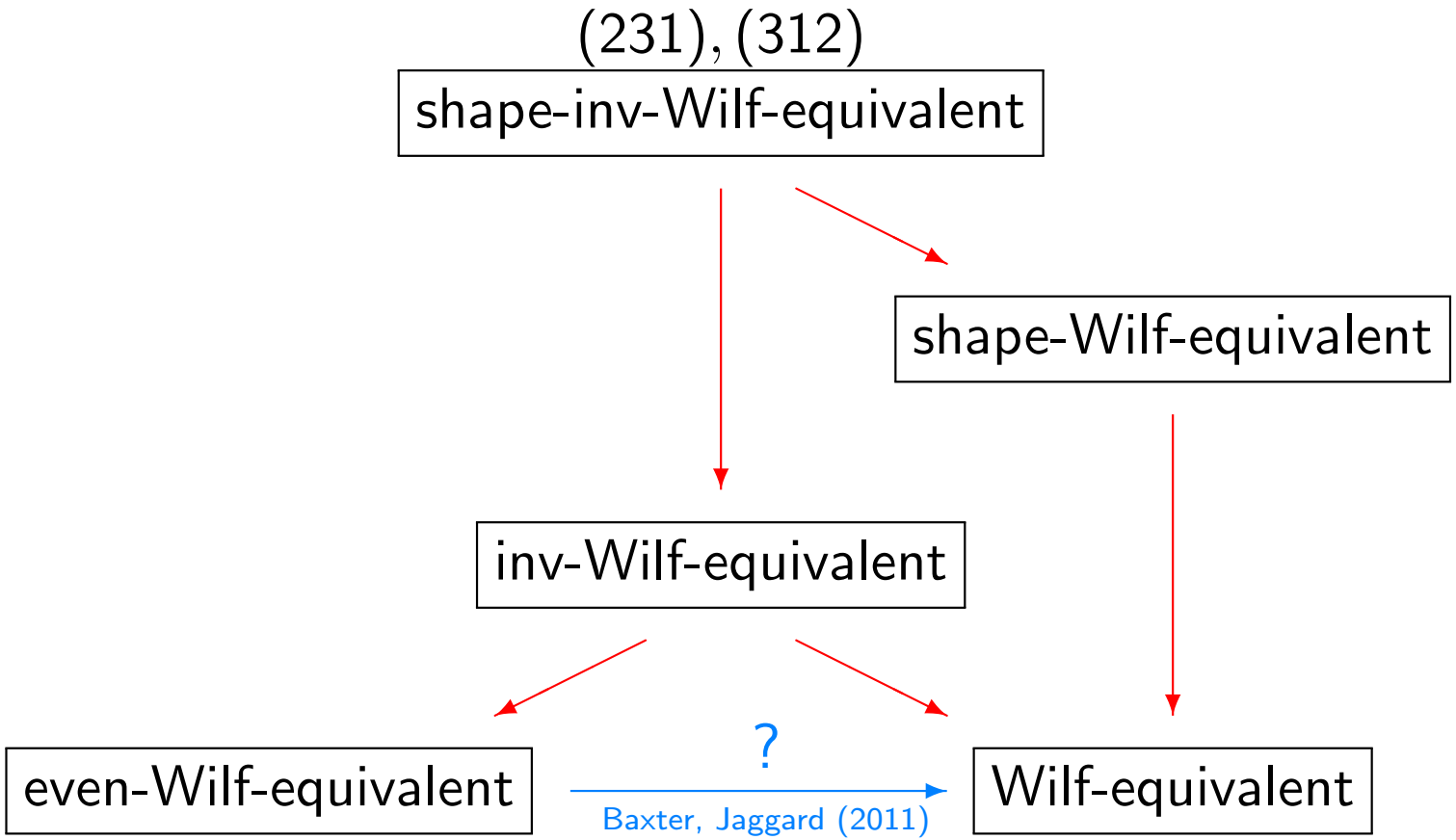
Inv-Wilf-equivalence implies even-Wilf-equivalence.


(231), (312)
shape-inv-Wilf-equivalent
shape-Wilf-equivalent
inv-Wilf-equivalent
Wilf-equivalent


(231), (312)
shape-inv-Wilf-equivalent
shape-Wilf-equivalent
inv-Wilf-equivalent
even-Wilf-equivalent
Wilf-equivalent


(231), (312)
shape-inv-Wilf-equivalent
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?
Baxter, Jaggard (2011)
Wilf-equivalent

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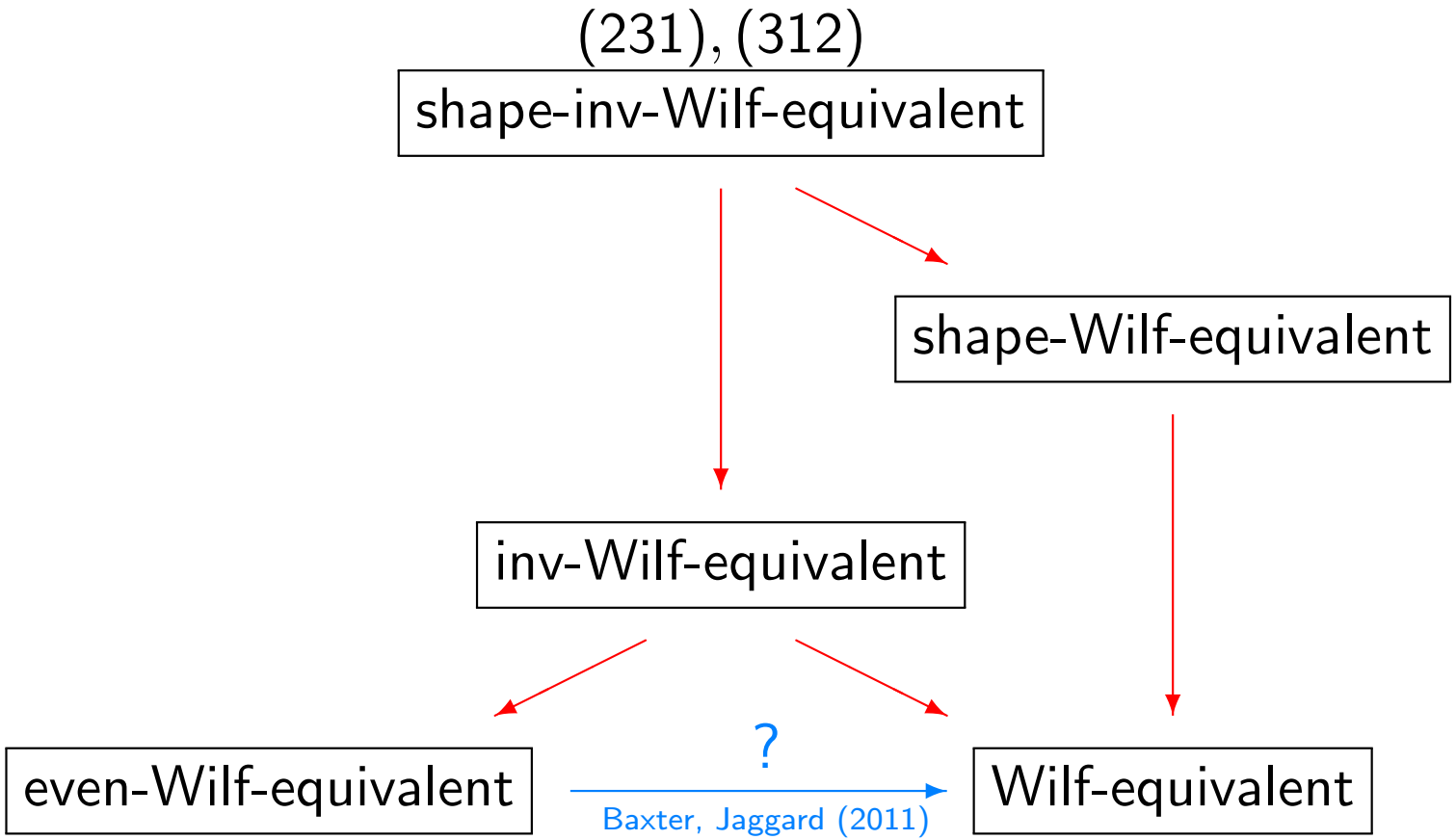
- Proving these even-Wilf-equivalences would complete the classification for length 6.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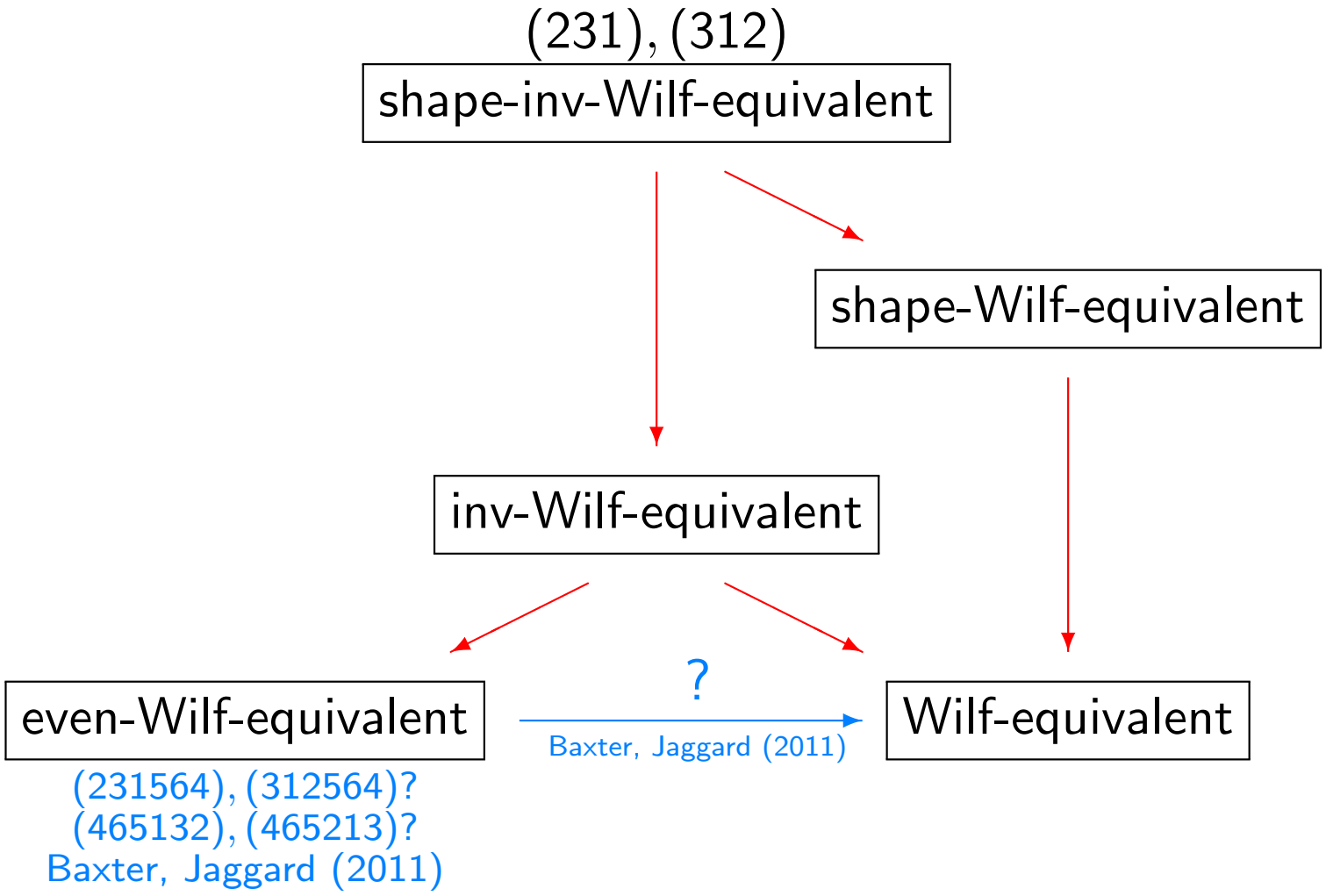
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Wilf-equivalent
Baxter, Jaggard (2011)
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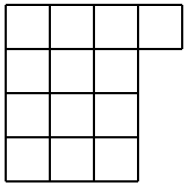
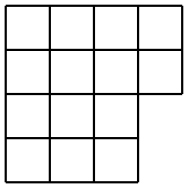
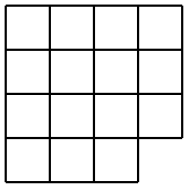
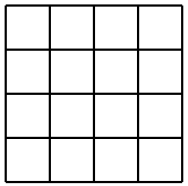
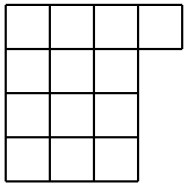
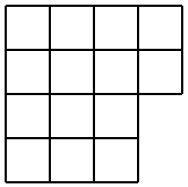
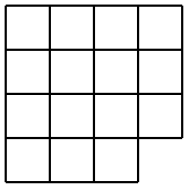
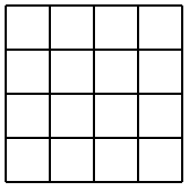
Baxter, Jaggard (2011)

$A_Y(\pi)$: Set of transversals with even inversion number in Young diagram Y that avoid π.

$A_Y(\pi)$: Set of transversals with even inversion number in Young diagram Y that avoid π.

Y	$\lvert A_Y(213)\rvert$	$\lvert A_Y(321)\rvert$
	2	2
	5	5
	6	6
	7	7

$A_Y(\pi)$: Set of transversals with even inversion number in Young diagram Y that avoid π.

Y	$\lvert A_Y(213)\rvert$	$\lvert A_Y(321)\rvert$
	2	2
	5	5
	6	6
	7	7

In fact, $|A_Y(213)| = |A_Y(321)|$ for all Young diagrams Y.

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

Pattern avoidance in permutations

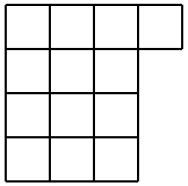
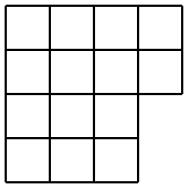
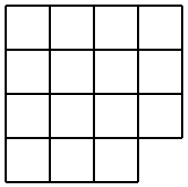
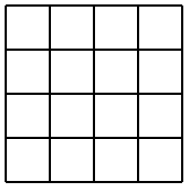
New result

Shape-Wilf-equivalence

Shape-inv-Wilf-equivalence

Even-Wilf-equivalence

$A_Y(\pi)$: Set of transversals with even inversion number in Young diagram Y that avoid π.

Y	$\lvert A_Y(213)\rvert$	$\lvert A_Y(321)\rvert$
	2	2
	5	5
	6	6
	7	7

In fact, $|A_Y(213)| = |A_Y(321)|$ for all Young diagrams Y. (213) and (321) are **even-shape-Wilf-equivalent**.

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

Pattern avoidance in permutations

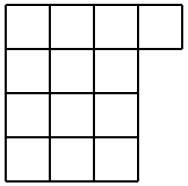
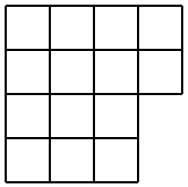
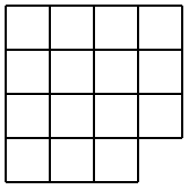
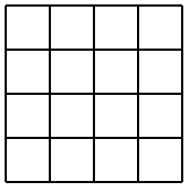
New result

Shape-Wilf-equivalence

Shape-inv-Wilf-equivalence

Even-Wilf-equivalence

$A_Y(\pi)$: Set of transversals with even inversion number in Young diagram Y that avoid π.

Y	$\lvert A_Y(213)\rvert$	$\lvert A_Y(321)\rvert$
	2	2
	5	5
	6	6
	7	7

In fact, $|A_Y(213)| = |A_Y(321)|$ for all Young diagrams Y.
(213) and (321) are **even-shape-Wilf-equivalent**.
Even-shape-Wilf-equivalence implies even-Wilf-equivalence.

An infinite family of inv-Wilf-equivalent permutation pairs

Justin Chan

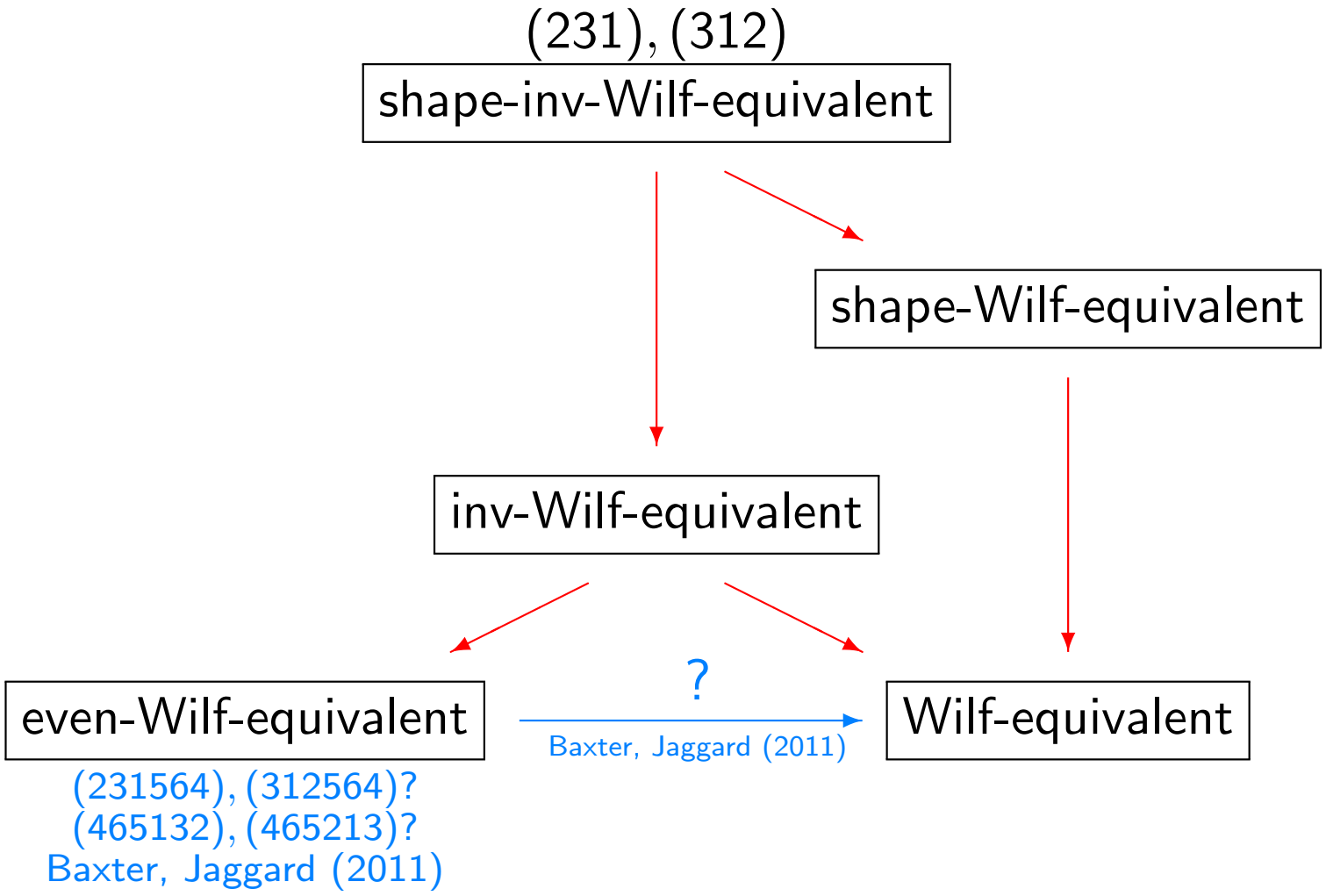
Pattern avoidance in permutations

New result

Shape-Wilf-equivalence

Shape-inv-Wilf-equivalence

Even-Wilf-equivalence


(231), (312)
shape-inv-Wilf-equivalent
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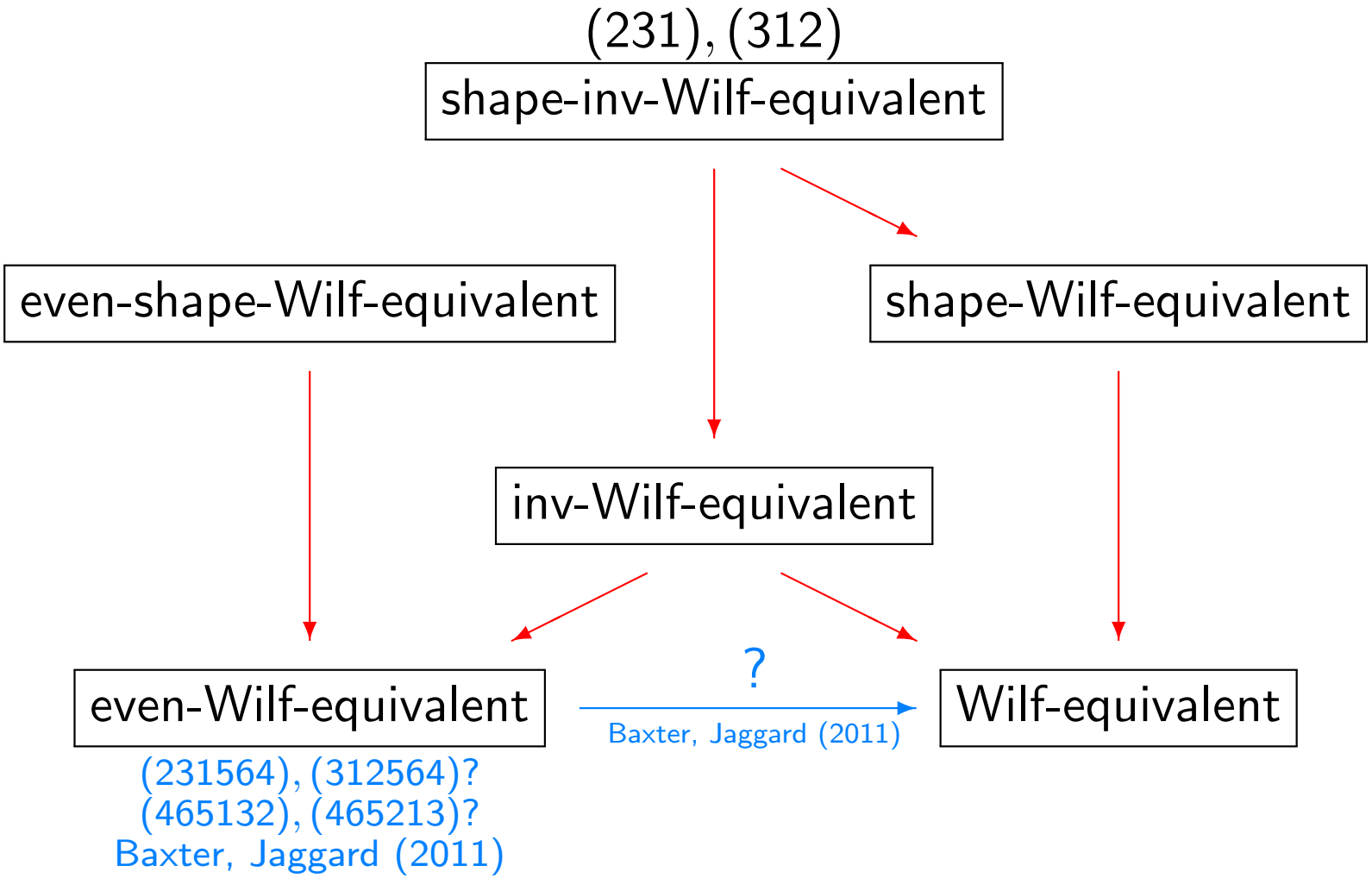
Baxter, Jaggard (2011)
(231564), (312564)?
(465132), (465213)?
Baxter, Jaggard (2011)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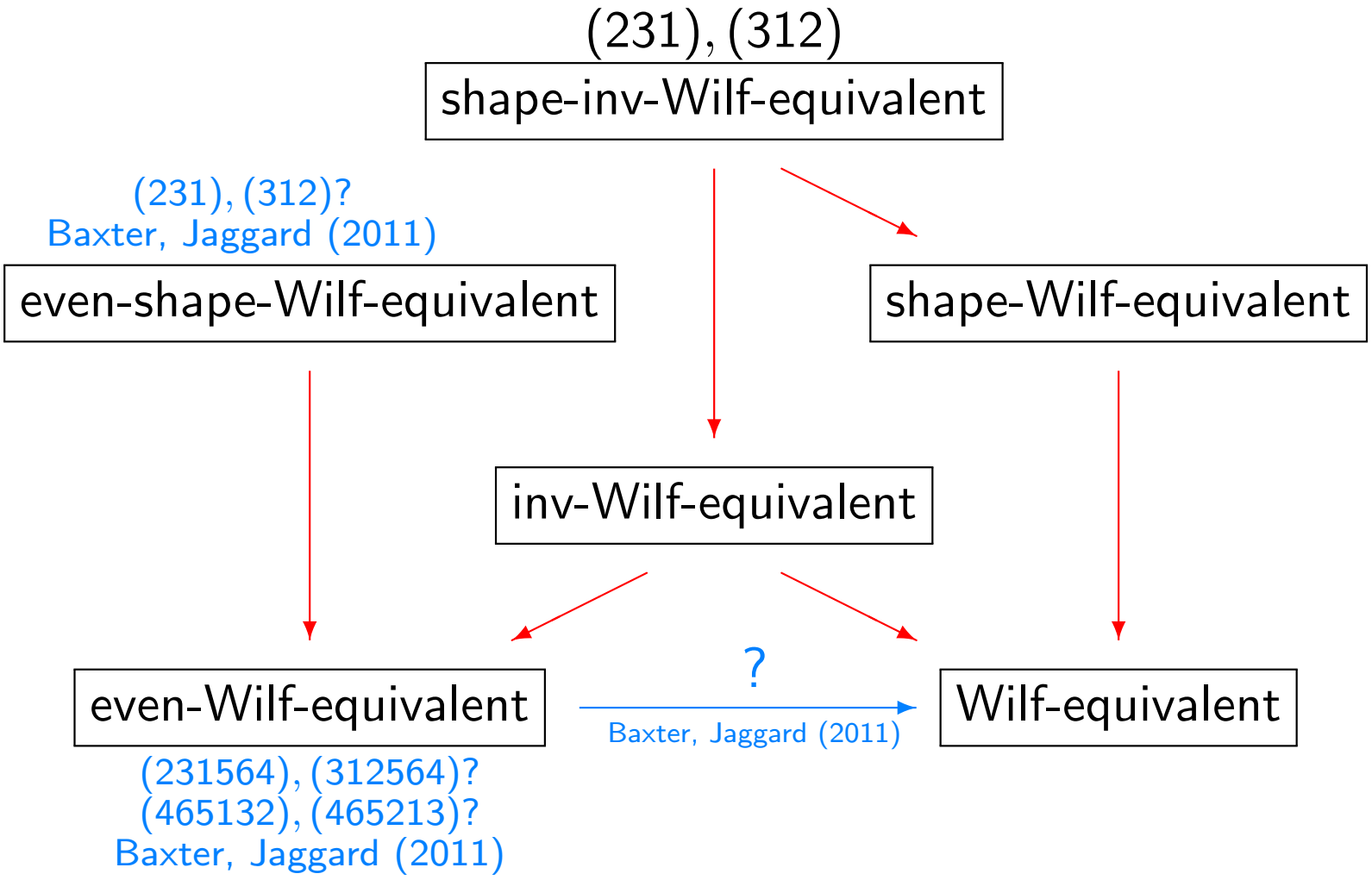
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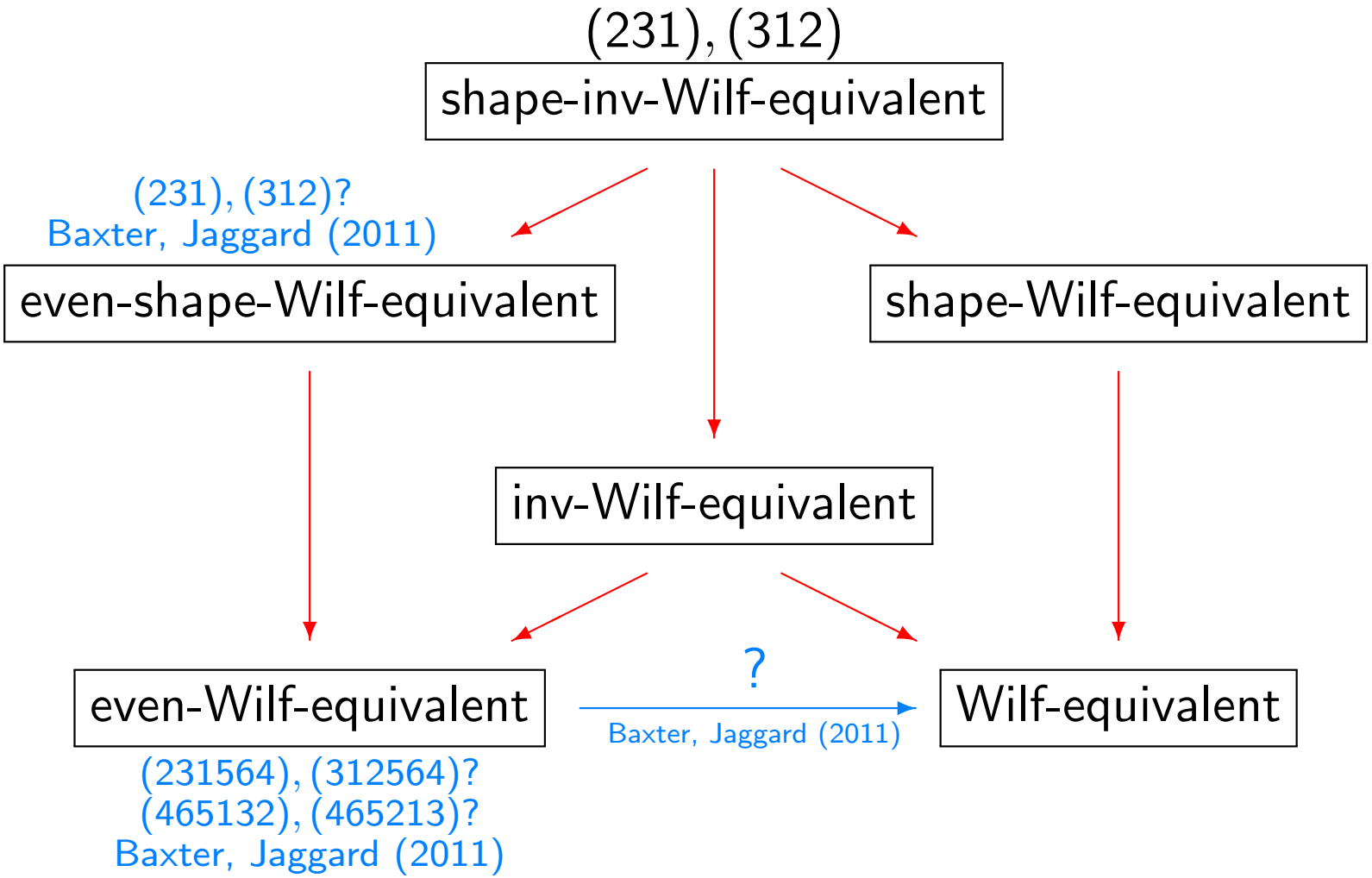
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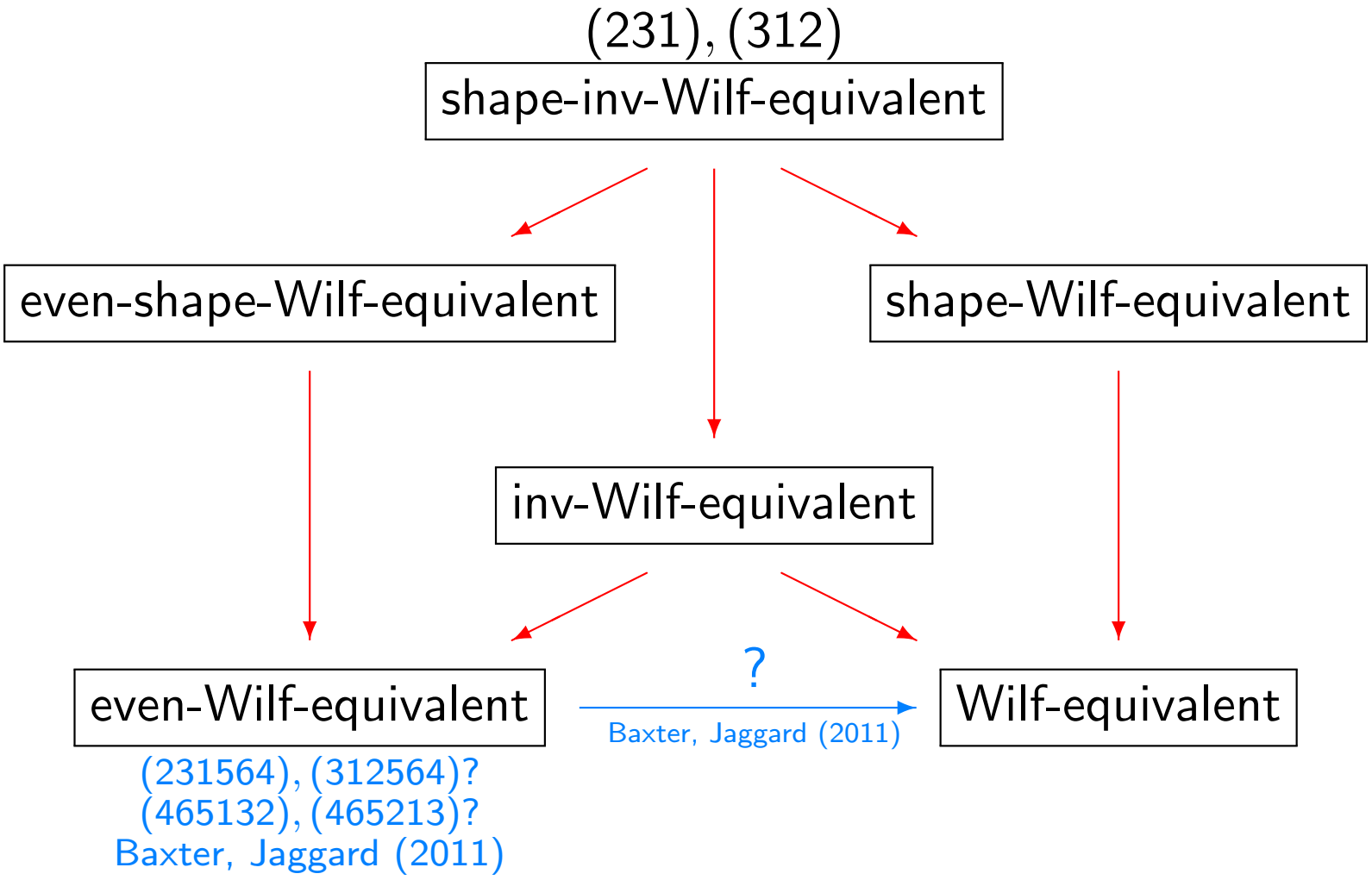
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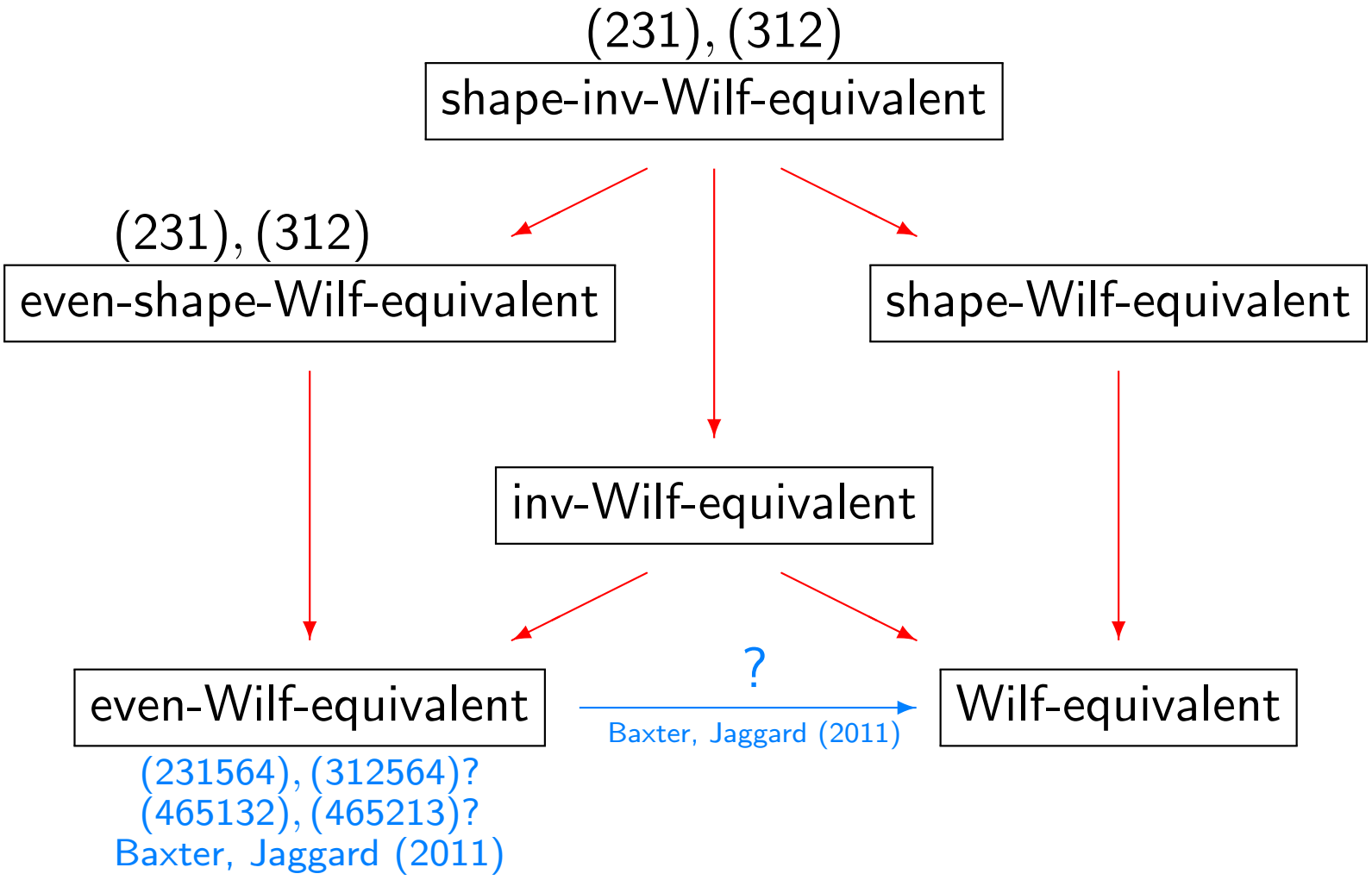
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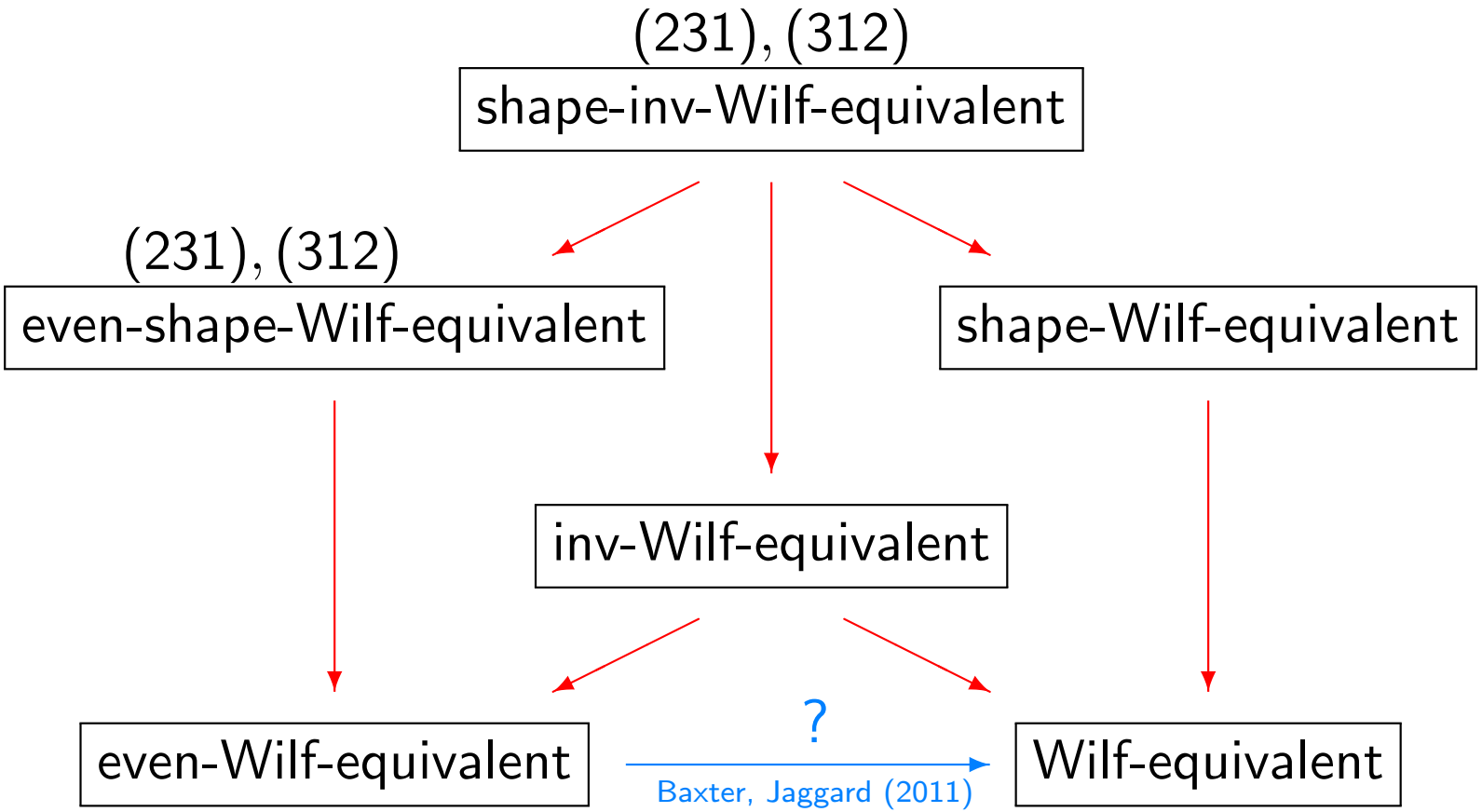
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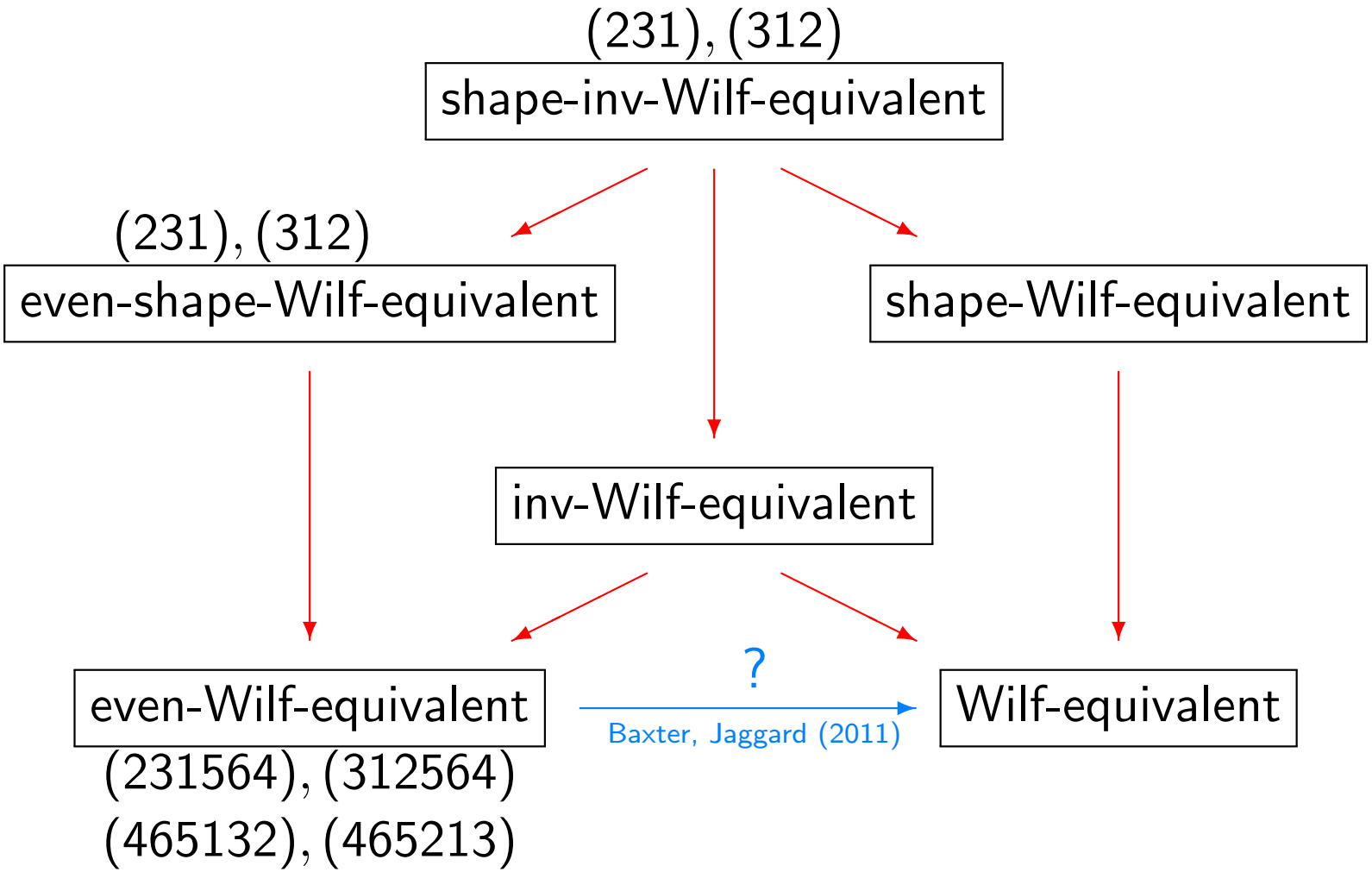
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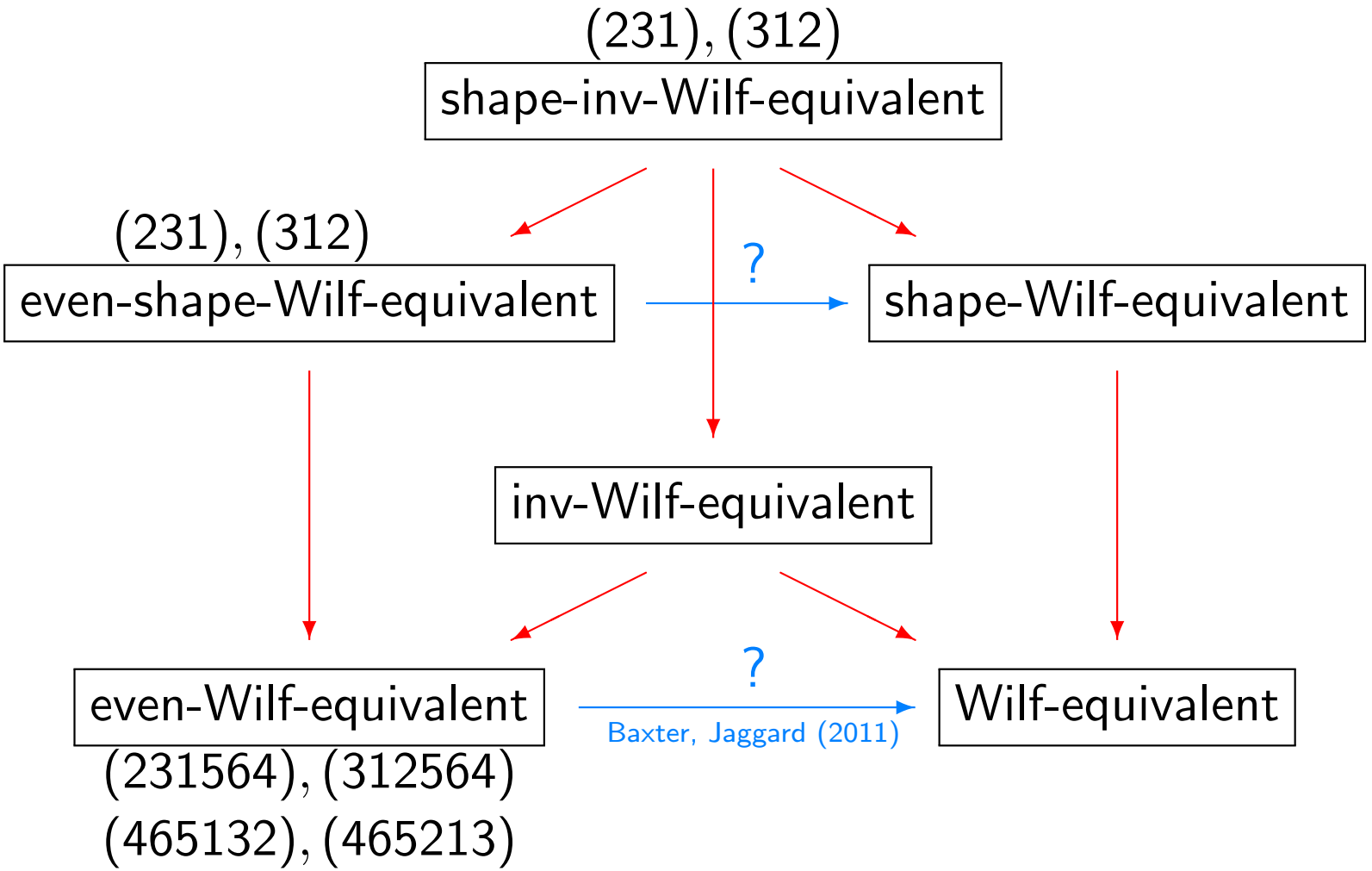
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