

(Permutations Arising from) Hook Coefficients of Chromatic Symmetric Functions

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Overview

- 1 Definition of a Chromatic Symmetric Function
- 2 Stanley's Construction
- 3 Patterns of Permutations

Colorings of a Graph, G

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Suppose that $G = P_3$, the path-3 graph. Let κ be the proper coloring:



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$$x^\kappa = x_1^2 x_2.$$

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The chromatic symmetric function evaluated in three variables is

$$X_{P_3}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 6x_1 x_2 x_3$$

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X_G is always symmetric and homogeneous of degree $n = |V|$. Recall that the ring of symmetric functions of degree n , Λ_n , forms a $\mathbb{Q}$ -vector space with various bases parameterized by partitions of n :

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Elementary Symmetric Functions:

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Schur Functions:

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T.$$

Quasisymmetric Functions

Construct the space of degree n quasisymmetric functions as the $\mathbb{Q}$ -vector space with basis parameterized by subsets of $[n - 1]$.

$$F_S = \prod_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_j = i_{j+1} \Rightarrow i_j \notin S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

These polynomials are called the (Gessel) fundamental quasisymmetric polynomials.

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Definition (Order-Preserving Map)

Given an acyclic orientation of G , an **order-preserving map** is a bijection $\phi : V \rightarrow \{1, \dots, n = |V|\}$ such that if uv is an arrow then $\phi(u) > \phi(v)$.

(Think “big numbers point to little numbers.”)

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The Permutations

Fix σ , an acyclic orientation of G . Select an order-reversing map ω_σ , and let A_σ be the set of all order-preserving maps defined on σ .

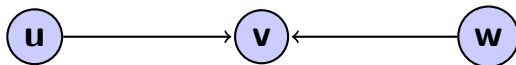
$\Omega_{\sigma, \omega_\sigma}$

For each $\alpha \in A_\sigma$, $\alpha\omega_\sigma^{-1}$ is necessarily a permutation. Define

$$\Omega_{\sigma, \omega_\sigma} = \{\alpha\omega_\sigma^{-1} \mid \alpha \in A_\sigma\}.$$

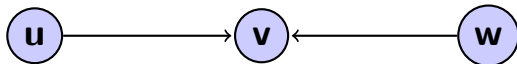
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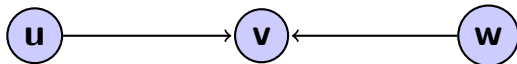


If we let $\omega = \begin{pmatrix} u & v & w \\ 1 & 3 & 2 \end{pmatrix}$ and

$$A = \left\{ \begin{pmatrix} u & v & w \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} u & v & w \\ 2 & 1 & 3 \end{pmatrix} \right\}$$

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then

$$\Omega_{0,\omega} = \{321, 231\} \text{ with inverse descent sets } \{\{1, 2\}, \{1\}\}.$$

Stanley's Construction

Theorem (Stanley, 1995)

Given the previous notation,

$$X_G = \sum_0 \sum_{\sigma \in \Omega_0, \omega_0} F_{ides(\sigma)},$$

where $ides(\sigma)$ is the set inverse descents of σ and F is the (Gessel) fundamental quasisymmetric function.

Stanley showed that the set of inverse descents

$$\{ides(\sigma) | \sigma \in \Omega_0, \omega_0\}$$

does not rely on the choice of ω_0 .

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Example: All $\sigma \in \Omega_{\mathfrak{o}, \omega_{\mathfrak{o}}}$ will avoid $123 \dots n$ unless G is the disjoint union of isolated points.

Example: If ω is an order-reversing map then exchanging labels j with $n - j$ results in an order-preserving map. Therefore a permutation containing $n \dots 321$ is always present in $\Omega_{\mathfrak{o}, \omega_{\mathfrak{o}}}$ for each acyclic orientation, $\mathfrak{o}$.

Application

We know that a permutation containing the pattern $n \dots 321$ always appears in $\Omega_{\mathbf{o}, \omega_{\mathbf{o}}}$ and $ides(n, n-1, \dots, 1) = [n-1]$. This gives us the following:

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Theorem (K. 2014)

If G is a graph then the coefficient of $F_{[n-1]} = s_{(1^n)}$ in X_G is the number of acyclic orientations of G .

Hook Coefficients of Clawfree Incomparability Graphs

If $\lambda^k \vdash n$ is the hook $(k, 1, \dots, 1)$ where $n = |V|$ then we have the following theorem.

Theorem (Chow, 1997)

If G is a clawfree incomparability graph and X_G is expanded in the basis of Schur functions then the hook coefficients are

$$c_{\lambda^k} = \sum_{j=1}^n \binom{j-1}{k-1} \cdot a_j,$$

where a_j is the number of acyclic orientations of G with precisely j sinks.

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Chow did *not* use Stanley's construction, but rather an argument involving P -tableaux.

Hook Coefficients of All Graphs

Using the same notation:

Theorem (K., 2014)

If G is any graph and X_G is expanded in the basis of Schur functions then the hook coefficients are

$$c_{\lambda^k} = \sum_{j=1}^n \binom{j-1}{k-1} \cdot a_j,$$

where a_j is the number of acyclic orientations of G with precisely j sinks.

Proof Sketch

How do we find the coefficient c_{λ^k} ?

We need to answer four questions.

Question 1

Which fundamental quasisymmetric functions correspond to the hook Schur functions?

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Answer: If

$$\begin{aligned} X_G &= \sum_{\alpha \models n} d_\alpha F_\alpha \\ &= \sum_{\lambda \vdash n} c_\lambda s_\lambda \end{aligned}$$

then $c_{\lambda^k} = d_{\lambda^k}$

[Egge, Loehr, Warrington 2010].

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There is a well-known correspondence between (inverse) descent sets and compositions. If $ides(\sigma) = \{i_1 < i_2 < \dots < i_t\}$ then the corresponding composition is $(i_1, i_2 - i_1, i_3 - i_2, \dots, n - i_t)$.

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- ① $\sigma^{-1}(k) = n$
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Example: Let $k = 4, n = 7$. One such permutation is $\sigma = (7126534)$.

Question 4

What “physical” phenomena occur in G that create permutations with the required inverse descent sets?

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Let s be the vertex such that $\omega(s) = n$. For all order-preserving maps α where $\alpha\omega^{-1}$ has the appropriate inverse descent set, $\alpha(s) = k$.

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Of the remaining $j - 1$ sinks choose $k - 1$ of them. If we insist that α maps the chosen sinks to values less than k then the conditions on $\sigma = \alpha\omega^{-1}$ completely determines the map α . Thus each acyclic orientation provides $\binom{j-1}{k-1}$ such maps.

What about near-hooks?

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- Permutations that have these inverse descent sets satisfy:
 - 1 $\sigma^{-1}(i) < \sigma^{-1}(i+1)$ for $i < k$
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 - ③ $\sigma^{-1}(k+1) < \sigma^{-1}(k), \sigma^{-1}(k+2)$
- What “physical” phenomena exist in a graph that correspond to permutations with these inverse descent sets?

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Thank You