

Patterns and Statistics for Restricted Growth Functions

Samantha Dahlberg

Robert Dorward

Jonathan Gerhard

Thomas Grubb

Carlin Purcell

Lindsey Reppuhn

Bruce E. Sagan

Acknowledgements

This work was performed with support from the Michigan State University REU program SURIEM, NSF grant 1062817, and NSA grant “RE-13-SURIEM-0613-msu-2-121015-Zeleke”.

Table of Contents

- 1 Introduction
 - Restricted Growth Functions vs Set Partitions
 - Statistics
- 2 General Results
 - RGFs of Length 3
- 3 Selected Results
 - $WRB_n(112)$
 - Multivariate Generating Functions
- 4 Closing Remarks
 - Future Work

Restricted Growth Functions (RGFs)

A *Restricted Growth Function* or *RGF* is a word $w = a_1 a_2 \dots a_n$ such that $a_1 = 1$ and

$$a_i \leq 1 + \max\{a_1, \dots, a_{i-1}\}.$$

Restricted Growth Functions (RGFs)

A *Restricted Growth Function* or *RGF* is a word $w = a_1 a_2 \dots a_n$ such that $a_1 = 1$ and

$$a_i \leq 1 + \max\{a_1, \dots, a_{i-1}\}.$$

For example, 12223412 is an RGF whereas 12433 is not.

Restricted Growth Functions (RGFs)

A *Restricted Growth Function* or *RGF* is a word $w = a_1 a_2 \dots a_n$ such that $a_1 = 1$ and

$$a_i \leq 1 + \max\{a_1, \dots, a_{i-1}\}.$$

For example, 12223412 is an RGF whereas 12433 is not.

An RGF w of length n corresponds to a set partition π such that $\pi \vdash [n]$.

RGF vs Set Partition Avoidance

Given an RGF w and another RGF v , which we call a *pattern*, we say that w *contains* the pattern if there is a subword $w' \subseteq w$ such that w' standardizes to v . Otherwise, we say w *avoids* v .

RGF vs Set Partition Avoidance

Given an RGF w and another RGF v , which we call a *pattern*, we say that w *contains* the pattern if there is a subword $w' \subseteq w$ such that w' standardizes to v . Otherwise, we say w *avoids* v .

For example, 12323 contains 112, because 223 standardizes to 112.

RGF vs Set Partition Avoidance

Given an RGF w and another RGF v , which we call a *pattern*, we say that w *contains* the pattern if there is a subword $w' \subseteq w$ such that w' standardizes to v . Otherwise, we say w *avoids* v .

For example, 12323 contains 112, because 223 standardizes to 112. On the other hand, 12343 avoids 112, because there is no such subword that standardizes to 112.

RGF vs Set Partition Avoidance

Given an RGF w and another RGF v , which we call a *pattern*, we say that w *contains* the pattern if there is a subword $w' \subseteq w$ such that w' standardizes to v . Otherwise, we say w *avoids* v .

For example, 12323 contains 112, because 223 standardizes to 112. On the other hand, 12343 avoids 112, because there is no such subword that standardizes to 112.

We denote the set of all RGFs of length n by $\mathcal{W}_n$. The *avoidance class* of v is defined to be

$$\mathcal{W}_n(v) = \{w \in \mathcal{W}_n \mid w \text{ avoids } v\}.$$

RGF vs Set Partition Avoidance

Given an RGF w and another RGF v , which we call a *pattern*, we say that w *contains* the pattern if there is a subword $w' \subseteq w$ such that w' standardizes to v . Otherwise, we say w *avoids* v .

For example, 12323 contains 112, because 223 standardizes to 112. On the other hand, 12343 avoids 112, because there is no such subword that standardizes to 112.

We denote the set of all RGFs of length n by $\mathcal{W}_n$. The *avoidance class* of v is defined to be

$$\mathcal{W}_n(v) = \{w \in \mathcal{W}_n \mid w \text{ avoids } v\}.$$

Sometimes the avoidance class of an RGF is identical to the words associated with the avoidance class of its associated set partition.

Often, however, this is not the case.

RGF vs Set Partition Avoidance

For example, words that avoid the word 112 strictly increase to a maximum m , then weakly decrease.

RGF vs Set Partition Avoidance

For example, words that avoid the word 112 strictly increase to a maximum m , then weakly decrease.

$$12 \dots m \ a_{m+1} \dots a_n$$

where $a_{m+1}, \dots, a_n \in [1, m]$, and $a_i \geq a_j, \forall m+1 \leq i < j \leq n$.

RGF vs Set Partition Avoidance

For example, words that avoid the word 112 strictly increase to a maximum m , then weakly decrease.

$$12 \dots m \ a_{m+1} \dots a_n$$

where $a_{m+1}, \dots, a_n \in [1, m]$, and $a_i \geq a_j, \forall m+1 \leq i < j \leq n$.

Words associated with set partitions that avoid 12/3 (whose RGF is 112) strictly increase to a maximum m , then are followed by a constant section of i 's such that $i \in [1, m]$.

RGF vs Set Partition Avoidance

For example, words that avoid the word 112 strictly increase to a maximum m , then weakly decrease.

$$12 \dots m \ a_{m+1} \dots a_n$$

where $a_{m+1}, \dots, a_n \in [1, m]$, and $a_i \geq a_j, \forall m+1 \leq i < j \leq n$.

Words associated with set partitions that avoid 12/3 (whose RGF is 112) strictly increase to a maximum m , then are followed by a constant section of i 's such that $i \in [1, m]$.

$$12 \dots m \ ii \dots i.$$

RGF vs Set Partition Avoidance

For example, words that avoid the word 112 strictly increase to a maximum m , then weakly decrease.

$$12 \dots m \ a_{m+1} \dots a_n$$

where $a_{m+1}, \dots, a_n \in [1, m]$, and $a_i \geq a_j, \forall m+1 \leq i < j \leq n$.

Words associated with set partitions that avoid 12/3 (whose RGF is 112) strictly increase to a maximum m , then are followed by a constant section of i 's such that $i \in [1, m]$.

$$12 \dots m \ ii \dots i.$$

This is because words associated with partitions that avoid 12/3 have to avoid 112, as well as 211, while simply avoiding the word 112 doesn't have this added restriction.

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(\textcolor{red}{1} \textcolor{blue}{2} \textcolor{blue}{3} \textcolor{blue}{3} \textcolor{blue}{2} \textcolor{blue}{4} \textcolor{blue}{5} \textcolor{blue}{4}) = \textcolor{red}{4} + 3 + 2 + 2 + 2 + 1 + 0 + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1 \text{ } 2 \text{ } 3 \text{ } 3 \text{ } 2 \text{ } 4 \text{ } 5 \text{ } 4) = 4 + 3 + 2 + 2 + 2 + 1 + 0 + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1 \ 2 \ 3 \ 3 \ 2 \ 4 \ 5 \ 4) = 4 + 3 + 2 + 2 + 2 + 1 + 0 + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1 \ 2 \ 3 \ \color{red}{3} \ 2 \ \color{blue}{4} \ \color{blue}{5} \ \color{blue}{4}) = 4 + 3 + 2 + \color{red}{2} + 2 + 1 + 0 + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1 \ 2 \ 3 \ 3 \ 2 \ 4 \ 5 \ 4) = 4 + 3 + 2 + 2 + 2 + 1 + 0 + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1 \ 2 \ 3 \ 3 \ 2 \ 4 \ 5 \ 4) = 4 + 3 + 2 + 2 + 2 + 1 + 0 + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1\ 2\ 3\ 3\ 2\ 4\ \textcolor{red}{5}\ 4) = 4 + 3 + 2 + 2 + 2 + 1 + \textcolor{red}{0} + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1\ 2\ 3\ 3\ 2\ 4\ 5\ 4) = 4 + 3 + 2 + 2 + 2 + 1 + 0 + 0$$

Statistics

We will use the four fundamental statistics on RGF's defined by Wachs and White: lb, ls, rb, and rs standing for “left-bigger,” “left-smaller,” “right-bigger,” and “right-smaller,” respectively.

For example, consider a restricted growth function $w = a_1 a_2 \dots a_n$.

$$\text{rb}(w) = \text{rb}(a_1) + \text{rb}(a_2) + \dots + \text{rb}(a_n).$$

where $\text{rb}(a_i)$ is the number of distinct integers to the right of a_i which are bigger than a_i . (Note that multiple copies of the same integer are only counted once). The three other statistics are defined in a comparable manner.

For example,

$$\text{rb}(1\ 2\ 3\ 3\ 2\ 4\ 5\ 4) = 4 + 3 + 2 + 2 + 2 + 1 + 0 + 0 = 14$$

Generating Functions for Statistics

We define the generating function $\text{WRB}_n(v)$ as:

$$\text{WRB}_n(v) = \sum_{w \in \mathcal{W}_n(v)} q^{rb(w)}.$$

Generating Functions for Statistics

We define the generating function $\text{WRB}_n(v)$ as:

$$\text{WRB}_n(v) = \sum_{w \in \mathcal{W}_n(v)} q^{\text{rb}(w)}.$$

Thus, the coefficient of q^k will be the number of words in the avoidance class of v that have an rb of k .

Generating Functions for Statistics

We define the generating function $\text{WRB}_n(v)$ as:

$$\text{WRB}_n(v) = \sum_{w \in \mathcal{W}_n(v)} q^{\text{rb}(w)}.$$

Thus, the coefficient of q^k will be the number of words in the avoidance class of v that have an rb of k .

We can define generating functions for all other statistics in a similar way.

Polynomials Generated from RGFs of Length 3

	Restricted Growth Functions				
Statistics	123	112	122	121	111
WLB	✓	✓	✓	✓	PC
WLS	✓	✓	✓	✓*	PC
WRB	✓	✓	✓	✓*	PC
WRS	✓	✓	✓	✓	PC

✓ = Completely Characterized

✓* = Already characterized in a previous paper by Sagan and Goyt.

PC = Partially Characterized

Gray Block = Same polynomial as corresponding Set Partition.

Polynomials Generated from RGFs of Length 3

$$\text{WLB}_n(122) = \text{WLB}_n(112) = \text{WRS}_n(112) = \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q.$$

$$\text{WLS}_n(122) = \text{WRB}_n(112) = \sum_{i=1}^n \binom{n-1}{i-1} q^{T_{i-1}}.$$

Polynomials Generated from RGFs of Length 3

$$\text{WLB}_n(122) = \text{WLB}_n(112) = \text{WRS}_n(112) = \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q.$$

$$\text{WLS}_n(122) = \text{WRB}_n(112) = \sum_{i=1}^n \binom{n-1}{i-1} q^{T_{i-1}}.$$

$$\text{WRB}_n(122) = \text{WLS}_n(112) = \prod_{i=1}^{n-1} (1 + q^i).$$

Polynomials Generated from RGFs of Length 3

$$\text{WLB}_n(122) = \text{WLB}_n(112) = \text{WRS}_n(112) = \sum_{m=0}^{n-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q.$$

$$\text{WLS}_n(122) = \text{WRB}_n(112) = \sum_{i=1}^n \binom{n-1}{i-1} q^{T_{i-1}}.$$

$$\text{WRB}_n(122) = \text{WLS}_n(112) = \prod_{i=1}^{n-1} (1 + q^i).$$

$$\text{WRS}_n(122) = 1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k.$$

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Proof.

Words that avoid 112 must be strictly increasing to some maximum element m , meaning our word must start with the sequence $12 \dots m$, and then weakly decrease afterwards. then weakly decreasing afterwards. Since we are looking at rb , only the elements that come before m will contribute, and their contribution will be

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Proof.

Words that avoid 112 must be strictly increasing to some maximum element m , meaning our word must start with the sequence $12 \dots m$, and then weakly decrease afterwards. then weakly decreasing afterwards. Since we are looking at rb, only the elements that come before m will contribute, and their contribution will be

$$\text{rb}(1 \ 2 \ 3 \ 4 \ \dots \ m-1 \ m) = \\ (m-1) + (m-2) + (m-3) + (m-4) + \dots + 1$$

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Proof.

Words that avoid 112 must be strictly increasing to some maximum element m , meaning our word must start with the sequence $12 \dots m$, and then weakly decrease afterwards. then weakly decreasing afterwards. Since we are looking at rb, only the elements that come before m will contribute, and their contribution will be

$$\text{rb}(1 \text{ } 2 \text{ } 3 \text{ } 4 \text{ } \dots \text{ } m-1 \text{ } m) = \\ (m-1) + (m-2) + (m-3) + (m-4) + \dots + 1$$

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Proof.

Words that avoid 112 must be strictly increasing to some maximum element m , meaning our word must start with the sequence $12 \dots m$, and then weakly decrease afterwards. then weakly decreasing afterwards. Since we are looking at rb, only the elements that come before m will contribute, and their contribution will be

$$\text{rb}(1 \ 2 \ 3 \ 4 \ \dots \ m-1 \ m) = \\ (m-1) + (m-2) + (m-3) + (m-4) + \dots + 1$$

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Proof.

Words that avoid 112 must be strictly increasing to some maximum element m , meaning our word must start with the sequence $12 \dots m$, and then weakly decrease afterwards. Then weakly decreasing afterwards. Since we are looking at rb , only the elements that come before m will contribute, and their contribution will be

$$\begin{aligned} \text{rb}(1 \ 2 \ 3 \ 4 \ \dots \ m-1 \ m) = \\ (m-1) + (m-2) + (m-3) + (m-4) + \dots + 1 \end{aligned}$$

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Proof.

Words that avoid 112 must be strictly increasing to some maximum element m , meaning our word must start with the sequence $12 \dots m$, and then weakly decrease afterwards. Since we are looking at rb, only the elements that come before m will contribute, and their contribution will be

$$\text{rb}(1 \ 2 \ 3 \ 4 \ \dots \ m-1 \ m) = \\ (m-1) + (m-2) + (m-3) + (m-4) + \dots + 1$$

Characterizing $\text{WRB}_n(112)$

Theorem (D,D,G,G,P,R,S)

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Proof.

Words that avoid 112 must be strictly increasing to some maximum element m , meaning our word must start with the sequence $12 \dots m$, and then weakly decrease afterwards. then weakly decreasing afterwards. Since we are looking at rb, only the elements that come before m will contribute, and their contribution will be

$$\begin{aligned} \text{rb}(1 \ 2 \ 3 \ 4 \ \dots \ m-1 \ m) &= \\ (m-1) + (m-2) + (m-3) + (m-4) + \dots + 1 &= T_{m-1}. \end{aligned}$$

Combinatorial Derivation of $\text{WRB}_n(112)$

Proof Continued.

So if your word has a maximum of m , its rb will be T_{m-1} .

Combinatorial Derivation of $\text{WRB}_n(112)$

Proof Continued.

So if your word has a maximum of m , its rb will be T_{m-1} .
After our maximum m , we must have a weakly decreasing sequence from the set $[m]$.

Combinatorial Derivation of $WRB_n(112)$

Proof Continued.

So if your word has a maximum of m , its rb will be T_{m-1} . After our maximum m , we must have a weakly decreasing sequence from the set $[m]$.

We are choosing $n - m$ such elements from $[m]$, allowing repetitions, and so we get

$$\left(\binom{m}{n-m} \right) = \binom{m+n-m-1}{n-m} = \binom{n-1}{m-1}.$$



Characterizing $\text{WRB}_n(112)$

So we get our final characterization,

$$\text{WRB}_n(112) = \sum_{m=1}^n \binom{n-1}{m-1} q^{T_{m-1}}.$$

Multivariate Generating Functions

After characterizing the words of length 3 for all four statistics, the obvious next step was looking at more than one statistic.

Multivariate Generating Functions

After characterizing the words of length 3 for all four statistics, the obvious next step was looking at more than one statistic.

To standardize notation, we have the following for a general generating function.

$$\text{WF}_n(v) = \sum_{w \in \mathcal{W}_n(v)} q^{\text{lb}(w)} r^{\text{ls}(w)} s^{\text{rb}(w)} t^{\text{rs}(w)} = \sum q^u r^x s^y t^z.$$

Multivariate Generating Functions

After characterizing the words of length 3 for all four statistics, the obvious next step was looking at more than one statistic.

To standardize notation, we have the following for a general generating function.

$$\text{WF}_n(v) = \sum_{w \in \mathcal{W}_n(v)} q^{\text{lb}(w)} r^{\text{ls}(w)} s^{\text{rb}(w)} t^{\text{rs}(w)} = \sum q^u r^x s^y t^z.$$

If we have a generating function for two or three statistics, they will be listed out as superscripts.

For example,

$$\text{WF}_n^{\text{lb}, \text{ls}}(v) = \sum_{w \in \mathcal{W}_n(v)} q^{\text{lb}(w)} r^{\text{ls}(w)} = \sum q^u r^x.$$

Generating Functions for Pairs of Statistics

$$\text{WF}_n^{\text{lb,rs}}(123) = n + \sum_{\substack{u, z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

Generating Functions for Pairs of Statistics

$$\text{WF}_n^{\text{lb,rs}}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

$$\text{WF}_n^{\text{ls,rb}}(123) = 1 + \sum_{\substack{x,y \geq 1 \\ x+y \leq n}} \binom{x+y-2}{y-1} r^x s^y.$$

Generating Functions for Pairs of Statistics

$$\text{WF}_n^{\text{lb,rs}}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

$$\text{WF}_n^{\text{ls,rb}}(123) = 1 + \sum_{\substack{x,y \geq 1 \\ x+y \leq n}} \binom{x+y-2}{y-1} r^x s^y.$$

$$\text{WF}_n^{\text{lb,ls}}(123) = \sum_{\substack{u,x \geq 1 \\ u+x \leq n}} \binom{u+x-1}{u} q^u r^x.$$

$WF_n^{lb,rs}(123)$

Theorem (D,D,G,G,P,R,S)

$$WF_n^{lb,rs}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

$WF_n^{lb,rs}(123)$

Theorem (D,D,G,G,P,R,S)

$$WF_n^{lb,rs}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

Proof.

We know that a word that avoids 123 must only have 1s and 2s.

$WF_n^{lb,rs}(123)$

Theorem (D,D,G,G,P,R,S)

$$WF_n^{lb,rs}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

Proof.

We know that a word that avoids 123 must only have 1s and 2s.

For some $w \in W_n(123)$, let's consider a word with $lb(w) = 0$.

$WF_n^{lb,rs}(123)$

Theorem (D,D,G,G,P,R,S)

$$WF_n^{lb,rs}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

Proof.

We know that a word that avoids 123 must only have 1s and 2s.

For some $w \in W_n(123)$, let's consider a word with $lb(w) = 0$.

If $lb(w) = 0$ then $rs(w) = 0$. The general form of such words is:

$11 \dots 12 \dots 2$.

$WF_n^{lb,rs}(123)$

Theorem (D,D,G,G,P,R,S)

$$WF_n^{lb,rs}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

Proof.

We know that a word that avoids 123 must only have 1s and 2s.

For some $w \in W_n(123)$, let's consider a word with $lb(w) = 0$.

If $lb(w) = 0$ then $rs(w) = 0$. The general form of such words is: $11 \dots 12 \dots 2$. The beginning one is forced, so there are $n-1$ choices of where to place the first two. Adding in the case where we have all 1s, we have n for our constant term.



$WF_n^{lb,rs}(123)$

Proof Continued..

We want to find out how many words have an $lb = u$ and an $rs = z$ where u and z are nonzero.

$WF_n^{lb,rs}(123)$

Proof Continued..

We want to find out how many words have an $lb = u$ and an $rs = z$ where u and z are nonzero.

To have the desired lb , we must have u 1s after our initial 2, and for the rs , we must have z 2s before our last one. This produces the general word:

$$1 \dots 1 \overbrace{2 \dots 1}^{u+z} \underbrace{2 \dots 2}_{u+z-2}$$

$WF_n^{lb,rs}(123)$

Proof Continued..

We want to find out how many words have an $lb = u$ and an $rs = z$ where u and z are nonzero.

To have the desired lb , we must have u 1s after our initial 2, and for the rs , we must have z 2s before our last one. This produces the general word:

$$1 \dots 1 \overbrace{2 \dots 1}^{u+z} \underbrace{2 \dots 2}_{u+z-2}$$

Since the 1 at the end of the sub-word is fixed, we then choose where to place the remaining $u - 1$ 1s in the available places to obtain $\binom{u+z-2}{u-1}$.

$WF_n^{lb,rs}(123)$

Proof Continued.

Since our sub-word takes up $u + z$ spaces, we may shift it around as many spaces as possible, namely $n - u - z$.

$WF_n^{lb,rs}(123)$

Proof Continued.

Since our sub-word takes up $u + z$ spaces, we may shift it around as many spaces as possible, namely $n - u - z$.

For example, consider the subword 21121, with $lb = 3$ and $rs = 2$. So $u + z = 5$.

$WF_n^{lb,rs}(123)$

Proof Continued.

Since our sub-word takes up $u + z$ spaces, we may shift it around as many spaces as possible, namely $n - u - z$.

For example, consider the subword 21121, with $lb = 3$ and $rs = 2$. So $u + z = 5$.

Suppose we want to place this subword in a word of length 8. Then we have $n - u - z = 8 - 5 = 3$ places to do so.

$WF_n^{lb,rs}(123)$

Proof Continued.

Since our sub-word takes up $u + z$ spaces, we may shift it around as many spaces as possible, namely $n - u - z$.

For example, consider the subword 21121, with $lb = 3$ and $rs = 2$. So $u + z = 5$.

Suppose we want to place this subword in a word of length 8. Then we have $n - u - z = 8 - 5 = 3$ places to do so.

11121121

$WF_n^{lb,rs}(123)$

Proof Continued.

Since our sub-word takes up $u + z$ spaces, we may shift it around as many spaces as possible, namely $n - u - z$.

For example, consider the subword 21121, with $lb = 3$ and $rs = 2$. So $u + z = 5$.

Suppose we want to place this subword in a word of length 8. Then we have $n - u - z = 8 - 5 = 3$ places to do so.

11121121

11211212

$WF_n^{lb,rs}(123)$

Proof Continued.

Since our sub-word takes up $u + z$ spaces, we may shift it around as many spaces as possible, namely $n - u - z$.

For example, consider the subword 21121, with $lb = 3$ and $rs = 2$. So $u + z = 5$.

Suppose we want to place this subword in a word of length 8. Then we have $n - u - z = 8 - 5 = 3$ places to do so.

11121121

11211212

1211212

$WF_n^{\text{lb,rs}}(123)$

Putting this together gives us the final polynomial:

$WF_n^{lb,rs}(123)$

Putting this together gives us the final polynomial:

$$WF_n^{lb,rs}(123) = n + \sum_{\substack{u,z \geq 1 \\ u+z \leq n-1}} \binom{u+z-2}{u-1} (n-u-z) q^u t^z.$$

THE Multivariate Generating Function for 123

Are You Ready?

THE Multivariate Generating Function for 123

$$\begin{aligned}
 \text{WF}_n(123) = & 1 + \sum_{\ell=1}^{n-1} \left[r^\ell s^{n-\ell} + \sum_{i=2}^{n-\ell} \left[q^{n-\ell-i+1} r^\ell s^{i-1} t^\ell + \right. \right. \\
 & \sum_{k=i+2}^n \binom{k-i-2}{n-i-\ell} q^{n-\ell-i+1} r^\ell s^{n-\ell} t^{\ell-n+k-1} + \\
 & \left. \left. \sum_{j=i+\ell}^{n-1} \binom{j-i-1}{j-\ell-i+1} q^{n-\ell-i+1} r^\ell s^{j-\ell} t^\ell \right] \right]
 \end{aligned}$$

Future Work

- Polynomials associated with RGF's of length four or greater.

Future Work

- Polynomials associated with RGF's of length four or greater.
- Multivariate Generating Functions.

Future Work

- Polynomials associated with RGF's of length four or greater.
- Multivariate Generating Functions.
- Other Statistics.

The End

Thank You!