

Enumeration of simple permutations in $\text{Av}(52341, 53241, 52431, 35142, 42513, 351624)$

Masaki Ikeda

University of Idaho

Permutation Patterns 2014

July 10, 2014

Before we start...

This work is still in progress for my doctoral dissertation. Proofs have not been carefully verified and in some parts still need some clarification.

Relation to Algebraic Geometry

Certain classes of Schubert varieties are indexed by permutations in some pattern avoidance classes.

Define

$$\mathcal{A} = \text{Av}(4231, 35142, 42513, 351624),$$

and

$$\mathcal{A}' = \text{Av}(52341, 53241, 52431, 35142, 42513, 351624).$$

Relation to Algebraic Geometry

Schubert varieties	Permutations
Smooth Schubert varieties	$\text{Av}(4231, 3412)$
Schubert varieties defined by inclusions	$\mathcal{A}$
Schubert varieties that are local complete intersections	$\mathcal{A}'$

Since the class $\mathcal{A}'$ is closely related to $\mathcal{A}$, we call permutations in $\mathcal{A}'$ *almost defined by inclusions*.

Enumeration of Each Class

Let $f_{\text{Av}(4231,3412)}$, $f_{\mathcal{A}}$ and $f_{\mathcal{A}'}$ be the generating functions which describes the numbers of length n permutations in $\text{Av}(4231,3412)$, $\mathcal{A}$ and $\mathcal{A}'$ respectively.

Generating Function for $\text{Av}(4231,3412)$ (M. Haiman, 1992)

$$f_{\text{Av}(4231,3412)} = \frac{x(1 - 5x + 4x^2 + x\sqrt{1 - 4x})}{1 - 6x + 8x^2 - 4x^3}$$

Generating Function for $\mathcal{A}$ (M. H. Albert, R. Brignall, 2013)

$$f_{\mathcal{A}} = \frac{1 - 3x - 2x^2 - (1 - x - 2x^2)\sqrt{1 - 4x}}{1 - 3x - (1 - x + 2x^2)\sqrt{1 - 4x}}$$

Enumeration of Each Class

The goal of my dissertation is to find $f_{\mathcal{A}'}$.

Enumeration of Each Class

The goal of my dissertation is to find $f_{\mathcal{A}'}$.

The approach we take to enumerate $\mathcal{A}'$ is similar to the method used by Albert and Brignall to enumerate $\mathcal{A}$.

- First, we investigate a structural characterization of simple permutations in $\mathcal{A}'$.
- Then we encode simple permutations into a regular language.
- Enumerate simple permutations in $\mathcal{A}$ using the transfer matrix method.
- Finally, we enumerate the whole class by inflation with some adjustments.

Simple Permutations

Example of inflation of a permutation:

$3\ 1\ 4\ 2[2\ 1, 1\ 3\ 2\ 4, 1\ 2, 1]$

Simple Permutations

Example of inflation of a permutation:

$$3 \underline{1} 4 2 [2 \underline{1}, \underline{1 3 2 4}, 1 2, 1]$$

Simple Permutations

Example of inflation of a permutation:

$$3 \underline{1} 4 2 [2 \underline{1}, \underline{1 3 2 4}, 1 \underline{2}, 1] = \underline{1 3 2 4}$$

Simple Permutations

Example of inflation of a permutation:

$$3\ 1\ 4\underline{2}[2\ 1, 1\ 3\ 2\ 4, 1\ 2, \underline{1}] = \quad 1\ 3\ 2\ 4$$

Simple Permutations

Example of inflation of a permutation:

$$3\ 1\ 4\ \underline{2}[2\ 1, 1\ 3\ 2\ 4, 1\ 2, \underline{1}] = \quad 1\ 3\ 2\ 4 \quad \underline{5}$$

Simple Permutations

Example of inflation of a permutation:

$$\underline{3} \, 1 \, 4 \, 2 [\underline{2} \, 1, 1 \, 3 \, 2 \, 4, 1 \, 2, 1] = \underline{7} \, 6 \, 1 \, 3 \, 2 \, 4 \quad 5$$

Simple Permutations

Example of inflation of a permutation:

$$3\ 1\ \underline{4}2[2\ 1, 1\ 3\ 2\ 4, \underline{1}2, 1] = 7\ 6\ 1\ 3\ 2\ 4\ \underline{8}9\ 5$$

Simple Permutations

Example of inflation of a permutation:

$$3\,1\,4\,2[2\,1,1\,3\,2\,4,1\,2,1] = 7\,6\,1\,3\,2\,4\,8\,9\,5$$

Simple Permutations

Definition

If the only way to obtain a permutation π by inflation is

$$\pi[1, 1, \dots, 1],$$

then we say a permutation π is simple.

Simple Permutations

Example/Non-example of Simple Permutations

simple
74153862

not simple
35417826

Simple Permutations

Example/Non-example of Simple Permutations

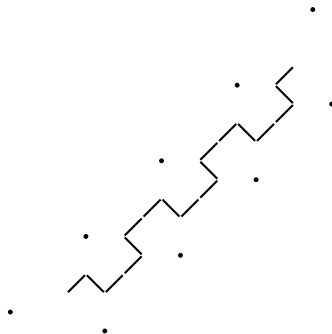
simple
74153862

not simple
35417826

$35417826 = 31524[132, 1, 12, 1, 1]$, so it is not simple.

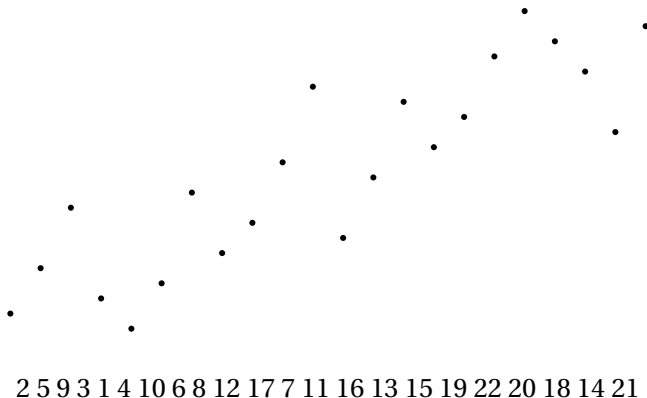
Structure of Simple Permutations in $\mathcal{A}$

Essentially, half of the simple permutations in $\mathcal{A}$ have the following structure:



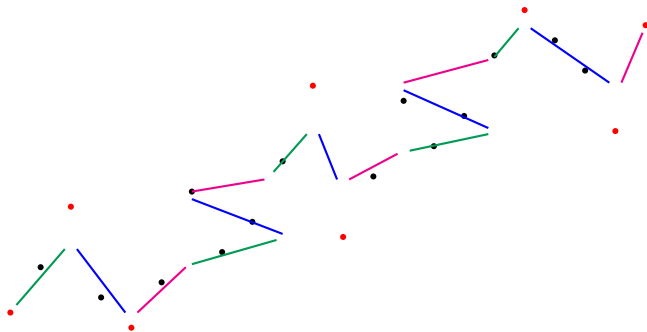
Structure of Simple Permutations in $\mathcal{A}$

Example of a permutation in $\mathcal{A}$ which has the described structure:



Structure of Simple Permutations in $\mathcal{A}$

Example of a permutation in $\mathcal{A}$ which has the described structure:



2 5 9 3 1 4 10 6 8 12 17 7 11 16 13 15 19 22 20 18 14 21

Encoding into a Regular Language

With the structural characterization we observed, we encode simple permutations in $\mathcal{A}$ into a regular language.

For each simple $\pi \in \mathcal{A}$, $\pi \mapsto w$ where w is a word over $\Sigma = \{a, b, c, d\}$.

Encoding into a Regular Language

With the structural characterization we observed, we encode simple permutations in $\mathcal{A}$ into a regular language.

For each simple $\pi \in \mathcal{A}$, $\pi \mapsto w$ where w is a word over $\Sigma = \{a, b, c, d\}$.

Example:

2 5 9 3 1 4 10 6 8 12 17 7 11 16 13 15 19 22 20 18 14 21

↓

d d b c a d c a b d c a d b a b d b a b d d

Finish up

- We now construct a finite state automaton accepting the language.
- Applying the transfer matrix method to the automaton, we find the generating function for the half of simple permutations in $\mathcal{A}$ to be

$$\frac{x^4}{(1-3x)(1+x)}.$$

- Inflation and some adjustments with this result gives us

$$f_{\mathcal{A}} = \frac{1-3x-2x^2-(1-x-2x^2)\sqrt{1-4x}}{1-3x-(1-x+2x^2)\sqrt{1-4x}}.$$

Conjecture of the Structure of Simple Permutations in $\mathcal{A}'$

We believe that the picture of half of simple permutations in $\mathcal{A}'$ is



So far, we observed the special case of simple permutations in $\mathcal{A}'$ closely.

Extreme Pattern of 2413

Definition

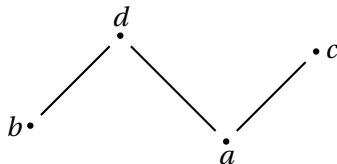
The extreme pattern of a permutation is the pattern defined by its first, last, greatest and least elements.

For example, a permutation $\underline{7}4\underline{1}53\underline{8}6\underline{2}$ has the extreme pattern 3142.

Here, we focus on the simple permutations in $\mathcal{A}'$ whose extreme pattern is 2413.

Extreme Pattern of 2413

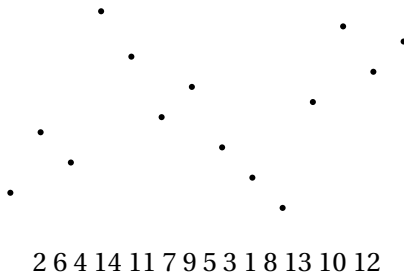
Revisiting $\mathcal{A}$, simple permutations of extreme pattern 2413 have an N-shaped structure.



Since $\mathcal{A} \subset \mathcal{A}'$, we study the picture of simple permutations of extreme pattern 2413 in $\mathcal{A}'$ which does not have the above structure.

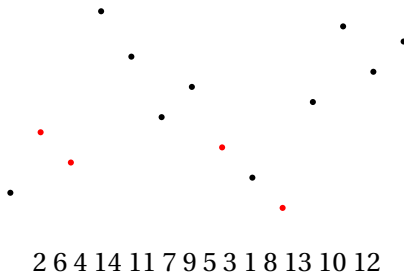
Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:



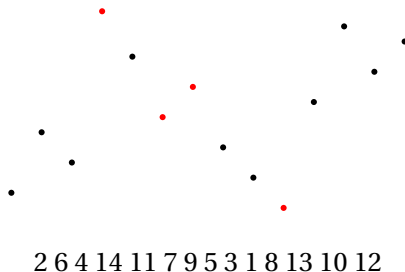
Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:



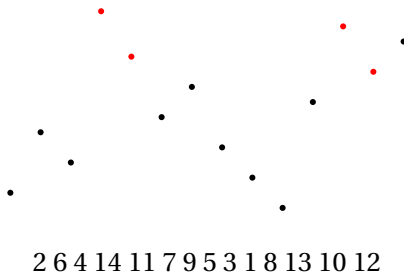
Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:



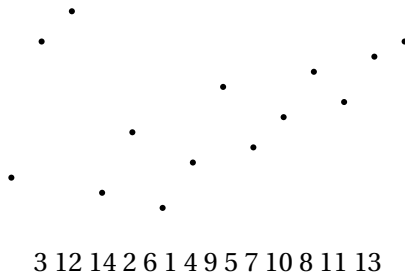
Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:



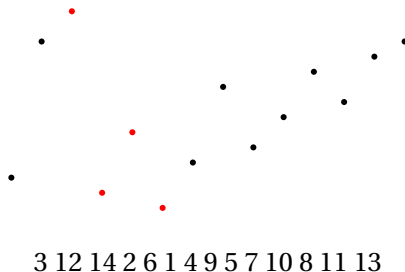
Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:



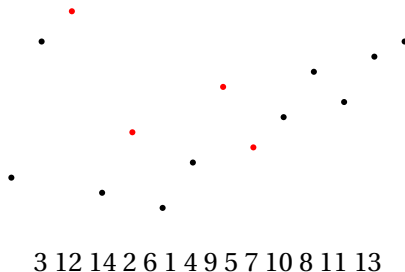
Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:



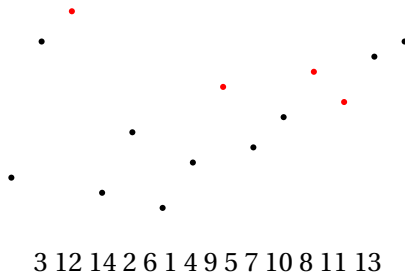
Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:

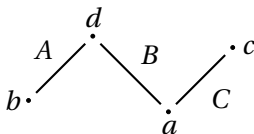
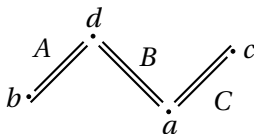


Extreme Pattern of 2413

Examples of simple permutations having extreme pattern 2413 that are in $\mathcal{A}' \setminus \mathcal{A}$:



Extreme Pattern of 2413



- In $\mathcal{A}$, intervals A and C are increasing, and B is decreasing. *I.e.*, A and C must avoid 21, while B must avoid 12. These are due to 4231 avoidance.
- In $\mathcal{A}'$, A and C must avoid 321, and B must avoid 123 instead, suggesting the structure above right.

Extreme Pattern of 2413

We now encode simple permutations of extreme pattern 2413 in $\mathcal{A}'$ into a regular language, which is an extension of the language that we used for $\mathcal{A}$.

Example:

$$2\ 6\ 4\ 14\ 11\ 7\ 9\ 5\ 3\ 1\ 8\ 13\ 10\ 12 \mapsto d\ d\ b\ d'\ b\ a''\ b'\ c\ b''\ c'\ b\ d\ c''\ d$$

$$3\ 12\ 14\ 2\ 6\ 1\ 4\ 9\ 5\ 7\ 10\ 8\ 11\ 13 \mapsto d\ b'\ d\ c\ c'\ b''\ c\ c'\ c''\ c''\ c\ a\ d\ d$$

Extreme Pattern of 2413

- Finally, we construct a finite state automaton accepting the language.
- With the transfer matrix method, we obtain the generating function for simple permutations of extreme pattern 2413 in $\mathcal{A}'$, which is

$$\frac{x^4(1 - x + 3x^2 + 6x^3 + 3x^4 + 3x^5 + 6x^6 + 5x^7 + 3x^8 + 3x^9 + x^{10})}{(1 + x)(1 - 3x - 2x^4 + 2x^5 + 9x^6 + 6x^7 - 4x^9 - 6x^{10} - 4x^{11} - x^{12})}.$$

- Starting from x^4 , coefficients of the power series are 1, 1, 8, 25, 79, 238, 724,...
- Details of the proof for this needs to be completed.

Extreme Pattern of 2413

- Finally, we construct a finite state automaton accepting the language.
- With the transfer matrix method, we obtain the generating function for simple permutations of extreme pattern 2413 in $\mathcal{A}'$, which is

$$\frac{x^4(1 - x + 3x^2 + 6x^3 + 3x^4 + 3x^5 + 6x^6 + 5x^7 + 3x^8 + 3x^9 + x^{10})}{(1 + x)(1 - 3x - 2x^4 + 2x^5 + 9x^6 + 6x^7 - 4x^9 - 6x^{10} - 4x^{11} - x^{12})}.$$

- Starting from x^4 , coefficients of the power series are 1, 1, 8, 25, 79, 238, 724,...
- Details of the proof for this needs to be completed.

Future Study

In order to enumerate $\mathcal{A}'$, I still have to do the following:

- Fix the generating function for the previous special case.
- Verify the conjecture about the structure of half of simple permutations in $\mathcal{A}'$.
- Come up with the extended encoding that works for half of simple permutations in $\mathcal{A}'$.
- Construct a finite state automaton accepting the language.
- Inflation process.
- Details of proofs for the results above.

Acknowledgements

- The use of *PermLab* has been extremely helpful to observe the structure of simple permutations in $\mathcal{A}'$.
- I would like to thank my advisor, Alexander Woo for the great support, and all the help towards my doctoral research as well as bringing my attention to Permutation Patterns 2014.

Acknowledgements

- The use of *PermLab* has been extremely helpful to observe the structure of simple permutations in $\mathcal{A}'$.
- I would like to thank my advisor, Alexander Woo for the great support, and all the help towards my doctoral research as well as bringing my attention to Permutation Patterns 2014.

Thank you for listening!