

A Reciprocity method and generating functions on words that avoid consecutive patterns

Jeff Remmel

Department of Mathematics
University of California, San Diego

Joint work with Luvreet Sangha (Ph.D. student UCSD).



Let $\mathbb{P} = \{1, 2, \dots\}$ denote the set of positive integers and let $[k] = \{1, \dots, k\}$ for any $k \in \mathbb{P}$. Let $\mathbb{P}^*$ ($[k]^*$) denote the set of all words over the alphabet $\mathbb{P}$ ($[k]$). We let ϵ denote the empty word and $\mathbb{P}^+ = \mathbb{P}^* - \{\epsilon\}$ and $[k]^+ = [k]^* - \{\epsilon\}$.

Given a word $w = w_1 \dots w_n \in \mathbb{P}^n$, we let $|w|$ denote the length of w , $z^w = z_{w_1} \dots z_{w_n}$, and

$$\begin{aligned} Des(w) &= \{i : w_i > w_{i+1}\} & Rise(w) &= \{i : w_i < w_{i+1}\} \\ WDes(w) &= \{i : w_i \geq w_{i+1}\} & WRise(w) &= \{i : w_i \leq w_{i+1}\} \\ des(w) &= |Des(w)| & ris(w) &= |Rise(w)| \\ wdes(w) &= |WDes(w)| & wris(w) &= |WRise(w)| \\ Lev(w) &= \{i : w_i = w_{i+1}\} & lev(w) &= |Lev(w)| \end{aligned}$$

For any $i, j \in \mathbb{P}$, we let $ij(w)$ equal the number of factors of the form ij in w .

Given a word $u \in \mathbb{P}^+$, we let $\text{red}(u)$ be the word that results by replacing each occurrence of the i -th smallest letter in u by i . For example, $\text{red}(2456582) = 1234351$.

If $u = u_1 \dots u_j \in \mathbb{P}^j$ and $w = w_1 \dots w_n \in \mathbb{P}^n$. Then

1. u **occurs** in w if there exists $1 \leq i_1 < \dots < i_j \leq n$ such that $\text{red}(w_{i_1} w_{i_2} \dots w_{i_j}) = u$.
2. w **avoids** u if u does not occur in w .
3. w has a **u -match starting at position i** if $\text{red}(w_i w_{i+1} \dots w_{i+j-1}) = u$.

We let $\text{umch}(w)$ equal the number of u matches in w .

1. u **exactly occurs** in w if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $w_{i_1} w_{i_2} \cdots w_{i_j} = u$.
2. w **exactly avoids** u if u does not exactly occur in w .
3. w has a **exact u -match starting at position i** if $w_i w_{i+1} \cdots w_{i+j-1} = u$.

We let $\text{eumch}(w)$ equal the number of u matches in w .

Let $\mathbf{z} = (z_1, z_2, \dots)$ and $u \in [k]^*$ such that $\text{red}(u) = u$. Let

$$N_{n,u}^{\mathbb{P}}(x, \mathbf{z}) = \sum_{w \in \mathbb{P}^n, \text{umch}(w)=0} x^{\text{des}(w)+1} z^w \text{ and}$$

$$N_{n,u}^{[k]}(x, \mathbf{z}) = \sum_{w \in [k]^n, \text{umch}(w)=0} x^{\text{des}(w)+1} z^w$$

and

$$\begin{aligned} \mathcal{N}_u^{\mathbb{P}}(x, \mathbf{z}, t) &= 1 + \sum_{w \in \mathbb{P}^+, \text{umch}(w)=0} x^{\text{des}(w)+1} z^w \\ &= 1 + \sum_{n \geq 1} N_{n,u}^{\mathbb{P}}(x, \mathbf{z}) t^n. \end{aligned}$$

$\mathcal{N}_u^{[k]}(x, \mathbf{z}, t)$ is defined similarly.

Goal. Find an explicit formula for $\mathcal{N}_u^{\mathbb{P}}(x, \mathbf{z}, t)$ and $\mathcal{N}_u^{[k]}(x, \mathbf{z}, t)$ in the case where $\text{des}(u) \leq 1$.

Let $u \in [k]^*$ and

$$EN_{n,u}^{\mathbb{P}}(x, \mathbf{z}) = \sum_{w \in \mathbb{P}^n, \text{eumch}(w)=0} x^{\text{des}(w)+1} z^w \text{ and}$$

$$EN_{n,u}^{[k]}(x, \mathbf{z}) = \sum_{w \in [k]^n, \text{eumch}(w)=0} x^{\text{des}(w)+1} z^w$$

and

$$\begin{aligned} \mathcal{EN}_u^{\mathbb{P}}(x, \mathbf{z}, t) &= 1 + \sum_{w \in \mathbb{P}^+, \text{eumch}(w)=0} x^{\text{des}(w)+1} z^w \\ &= 1 + \sum_{n \geq 1} N_{n,u}^{\mathbb{P}}(x, \mathbf{z}) t^n. \end{aligned}$$

$\mathcal{EN}_u^{[k]}(x, \mathbf{z}, t)$ is defined similarly.

Elementary & homogeneous symmetric functions

The n^{th} elementary symmetric function e_n is defined by

$$E(t) = \sum_{n \geq 0} e_n t^n := \prod_i (1 + x_i t).$$

The n^{th} homogeneous symmetric function h_n is defined by

$$H(t) = \sum_{n \geq 0} h_n t^n := \prod_i \frac{1}{1 - x_i t}.$$

The n^{th} power symmetric function p_n is defined by

$$p_n = \sum_i x_i^n.$$

If $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a partition, then

$$h_\lambda = \prod_{i=1}^k h_{\lambda_i}, \quad e_\lambda = \prod_{i=1}^k e_{\lambda_i}, \quad \text{and} \quad p_\lambda = \prod_{i=1}^k p_{\lambda_i},$$

Let Λ denote the ring of symmetric functions over infinitely many variables $x_1, x_2, \dots$. It is well known that $\{e_\lambda : \lambda \text{ a partition}\}$ is a basis for the ring of symmetric functions Λ . Thus $e_0, e_1, \dots$ is an algebraically independent set of generators for Λ .

Thus to define a ring homomorphism $\theta : \Lambda \rightarrow R$ into some ring R , we need only define $\theta(e_n)$ for $n \geq 0$.

Let $B_{\lambda,n}$ be the number of brick tabloids of shape (n) and type λ :

⋮ ⋮ ⋮	⋮ ⋮ ⋮		⋮ ⋮ ⋮	⋮ ⋮ ⋮
-------------	-------------	--	-------------	-------------

Theorem 1. (*Eğecioğlu-Remmel 1991*)

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_{\lambda}.$$

A link from Λ to enumeration problems in words

Assume that

$$\mathcal{N}_u^{\mathbb{P}}(x, \mathbf{z}, t) = 1 + \sum_{n \geq 1} N_{n,u}^{\mathbb{P}}(x, \mathbf{z}) t^n = \frac{1}{1 + \sum_{n \geq 1} U_{n,u}(x, \mathbf{z}) t^n}.$$

Then

$$1 + \sum_{n \geq 1} U_{n,u}(x, \mathbf{z}) t^n = \frac{1}{1 + \sum_{n \geq 1} N_{n,u}^{\mathbb{P}}(x, \mathbf{z}) t^n} \quad (1)$$

Define a ring homomorphism $\theta : \Lambda \rightarrow Q[x]$ by $\theta(e_0) = 1$ and

$$\theta(e_n) = (-1)^n N_{n,u}^{\mathbb{P}}(x, \mathbf{z}) \text{ for } n \geq 1.$$

Then

$$\begin{aligned} \theta(H(t)) &= 1 + \sum_{n \geq 1} \theta(h_n) \\ &= \theta(1/E(-t)) = \frac{1}{1 + \sum_{n \geq 1} \theta(e_n)(-t)^n} \\ &= \frac{1}{1 + \sum_{n \geq 1} N_{n,u}^{\mathbb{P}}(x, \mathbf{z})t^n} \\ &= 1 + \sum_{n \geq 1} U_{n,u}(x, \mathbf{z})t^n. \end{aligned}$$

Therefore

$$\theta(h_n) = U_{n,u}(x, \mathbf{z}).$$

$$\begin{aligned}
 \theta(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \theta(e_\lambda) \\
 &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i} N_{\lambda_i,u}^{\mathbf{P}}(x, \mathbf{z}) \\
 &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} N_{b_i,u}^{\mathbf{P}}(x, \mathbf{z}).
 \end{aligned}$$

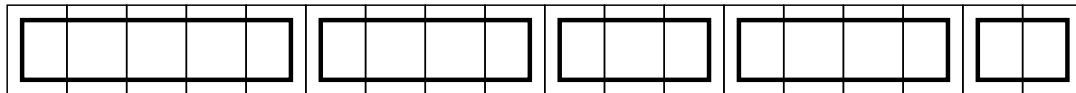
We have

$$\theta(h_n) = \sum_{\lambda \vdash n} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda, n}} (-1)^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} N_{b_i, u}^{\mathbb{P}}(x, \mathbf{z})$$

from which we create the following objects:

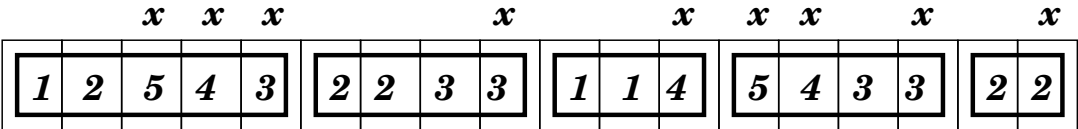
We have

$$\theta(h_n) = \sum_{\lambda \vdash n} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda, n}} (-1)^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} N_{b_i, u}^{\mathbb{P}}(x, \mathbf{z})$$



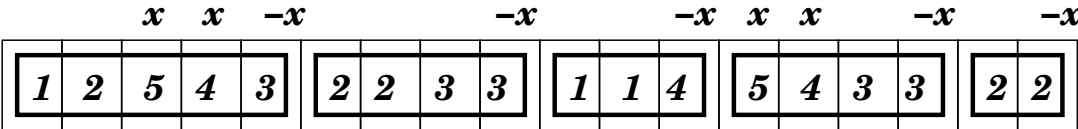
$$\theta(h_n) = \sum_{\lambda \vdash n} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda, n}} (-1)^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} N_{b_i, u}^{\mathbb{P}}(x, \mathbf{z})$$

$u = 2211$



$$\theta(h_n) = \sum_{\lambda \vdash n} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda, n}} (-1)^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} N_{b_i, u}^{\mathbb{P}}(x, \mathbf{z})$$

$u = 2211$



Thus $\theta(h_n)$ is the set of triples (B, w, L) such that

1. $B = (b_1, \dots, b_k)$ is a brick tabloid of length n ,
2. $w = w_1 \dots w_n \in \mathbb{P}^n$ is a word such that with in each brick b_i , there are no u -matches, and
3. $L : \{1, \dots, n\} \rightarrow \{1, x, -x\}$ is a labeling such that
 $L(i) = x$ if $w_i > w_{i+1}$ and w_i and w_{i+1} lie in the same brick,
 $L(i) = -x$ if i is last cell of some brick b_j , and
 $L(i) = 1$, otherwise.

Let $\mathcal{F}_n$ denote the set of all triples (B, w, L) satisfying (1)-(3).

Define the weight of (B, w, L) by $wt(B, w, L) = z^w \prod_{i=1}^n L(i)$.

Then

$$\theta(h_n) = \sum_{(B, w, L) \in \mathcal{F}_n} wt(B, w, L).$$

Next we define an involution $I_n : \mathcal{F}_n \rightarrow \mathcal{F}_n$ such that

(i) If $I_n(B, w, L) = (B', w', L') \neq (B, w, L)$, then $wt(B, w, L) = -wt(B', w', L')$ and

(ii) $I_n(B, w, L) = (B, w, L)$, then $wt(B, w, L) = x^k z^w$ for some $k \geq 1$.

Let $Fix(I_n)$ denote the set of fixed points of I_n .

It will then follow that

$$\theta(h_n) = \sum_{(B, w, L) \in Fix(I_n)} wt(B, w, L).$$

Definition of I_n on $(B = (b_1, \dots, b_l), w, L)$.

Look for the first cell c , reading from left to right, such that either

- (i) c is labeled with x or
- (ii) c is at the end of a brick b_i such that $i < k$, $w_c > w_{c+1}$, and there is no u -match in the cells of b_i and b_{i+1} .

In case (i), suppose that c is in brick b_i . Then

$$I_n(B, w, L) = (b_1, \dots, b_{i-1}, b^*, b^{**}, b_{i+1}, \dots, b_k), w, L')$$

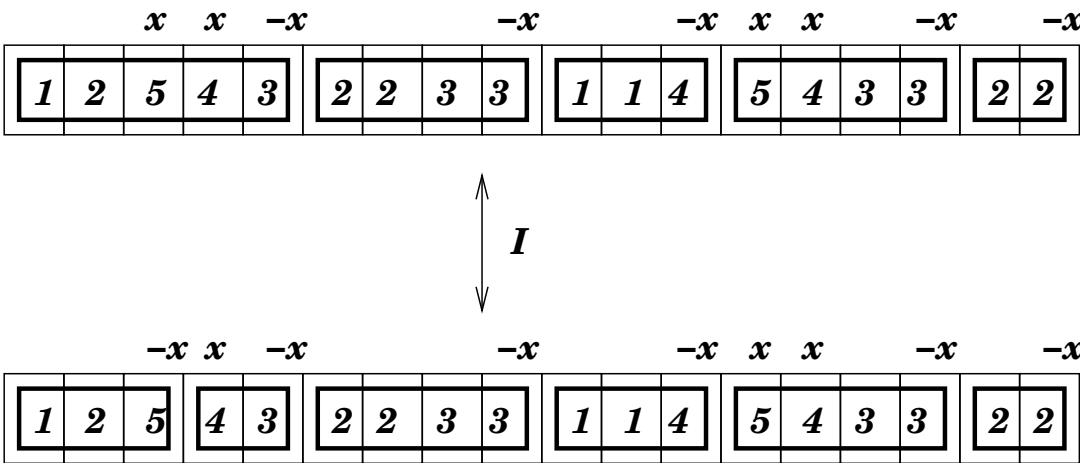
where b^* contains the cells of b_i up to and including cell c and b^{**} contains the rest of cells and $L'(i) = L(i)$ if $i \neq c$ and $L'(c) = -x$.

In case (ii),

$$I_n((B, w, L) = (b_1, \dots, b_{i-1}, b, b_{i+2}, \dots, b_k), w, L')$$

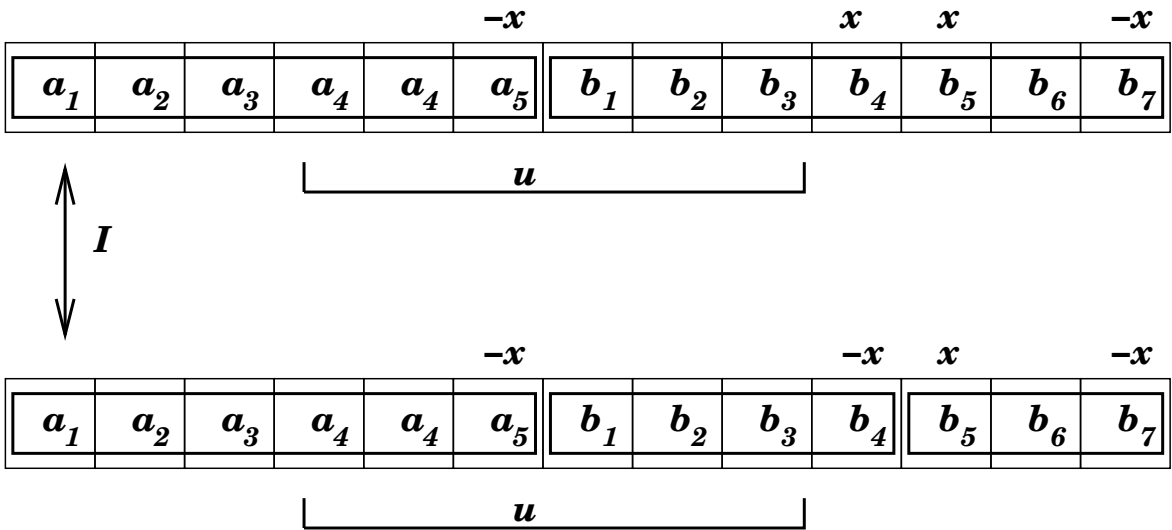
where b is the brick that results by combining b_i and b_{i+1} into a single brick and $L'(i) = L(i)$ if $i \neq c$ and $L'(c) = x$.

$u = 2211$



Claim. If $des(u) = 1$, then I_n is an involution

$u = 124345$



The fixed points of I_n .

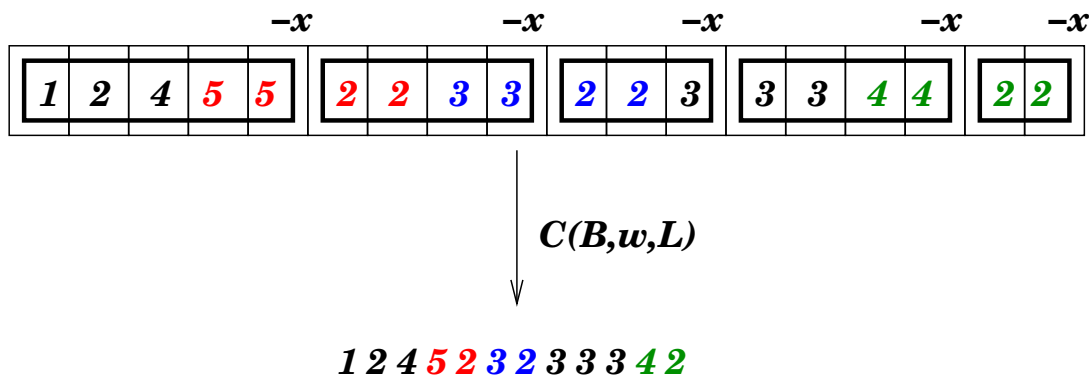
Given $(B, w, L) \in \mathcal{F}_n$, if $B = (b_1, \dots, b_k)$ and $1 \leq i \leq k$, we let $\text{last}(b_i)$ be the letter in the right most cell of b_i and $\text{first}(b_i)$ be the letter in the left most cell of b_i .

Then $(B, w, L) \in \text{Fix}(I_n)$ if and only if

1. There are no x labels which implies that w is weakly increasing within bricks.
2. For any two consecutive bricks b_i and b_{i+1} in B either
 - (i) $\text{last}(b_i) \leq \text{first}(b_{i+1})$ or
 - (ii) $\text{last}(b_i) > \text{first}(b_{i+1})$ and there is a u -match in the cells of b_i and b_{i+1} which must necessary involve $\text{last}(b_i)$ and $\text{first}(b_{i+1})$.

The Collapse Map

$$u = 2211$$



Theorem 2. *Let $u = 2^r 1^s$ where $r, s \geq 1$ and $r + s \geq 3$. Let $v = v_1 \dots v_p \in \mathbb{P}^+$. Then*

$$\sum_{n \geq p} t^n \sum_{\substack{(B, w, L) \in \text{Fix}(I_n) \\ C(B, w, L) = v}} \text{wt}(B, w, L) =$$

$$- x z^v t^p \left((1 - x)^{\text{wris}(w)} \prod_{i < j} x_{ji}^{\text{ji}(v)} \Big|_{x_{ji} = -x t^{r+s-2} z_i^{s-1} z_j^{r-1}} \right).$$

Theorem 3. *Let $u = 2^r 1^s$ where $r, s \geq 1$ and $r + s \geq 3$. Then*

$$\sum_{n \geq 1} U_n(x, \mathbf{z}) t^n =$$

$$- x \sum_{v \in \mathbb{P}^+} z^v t^{|v|} (1 - x)^{\text{wris}(w)} \left(\prod_{i < j} x_{ji}^{\text{ji}(v)} \Big|_{x_{ji} = -x t^{r+s-2} z_i^{s-1} z_j^{r-1}} \right).$$

For any word $w = w_1 \dots w_n \in \mathbb{P}^n$, we let $z^w = z_{w_1} \dots z_{w_n}$

Theorem 4. (*Fuller-Remmel*)

$$1 + \sum_{n \geq 1} t^n \sum_{w \in \mathbb{P}^n} x^{\text{des}(w)} z^w = \frac{1 - x}{-x + \prod_{j \geq 1} (1 - t(x - 1)z_j)} \quad (2)$$

Using the substitution $t \rightarrow ty$ and $x \rightarrow x/y$ and some algebra, one finds that

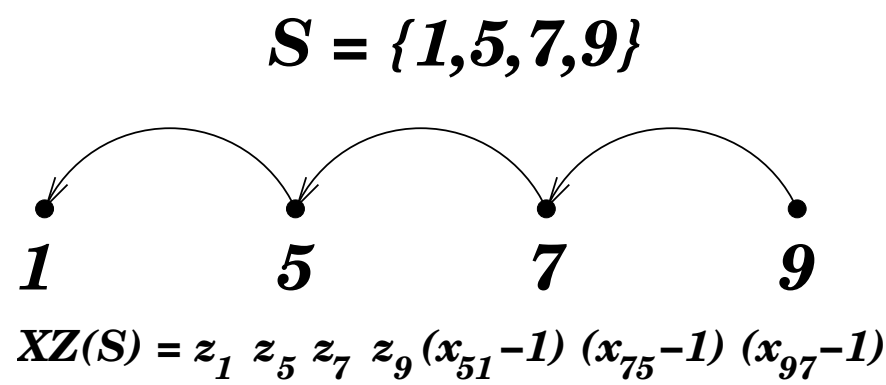
$$\sum_{n \geq 1} t^n \sum_{w \in \mathbb{P}^n} x^{\text{des}(w)} y^{\text{wris}(w)} z^w = \frac{1 - \prod_{j \geq 1} (1 + t(x - y)z_j)}{-x + y \prod_{j \geq 1} (1 + t(x - y)z_j)}. \quad (3)$$

Let

$$\mathcal{F}^{\mathbb{P}}(\mathbf{x}, \mathbf{z}, t) := \sum_{w \in \mathbb{P}^*} t^{|w|} z^w \prod_{i < j} x_{ji}^{\text{ji}(w)}.$$

Given a set $S \subseteq \mathbb{P}$, we let

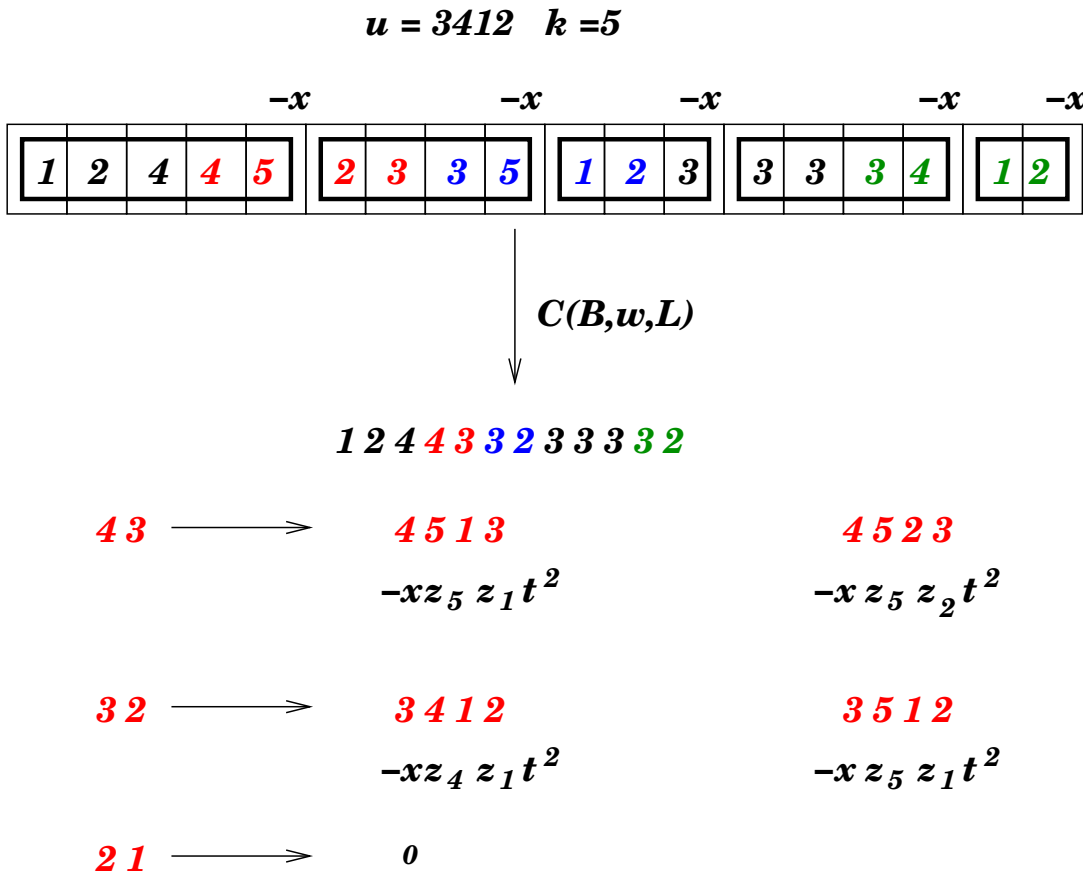
$$XZ(S) = \begin{cases} -1 & \text{if } S = \emptyset, \\ z_j & \text{if } S = \{j\}, \text{ and} \\ z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1}j_i} - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2. \end{cases}$$



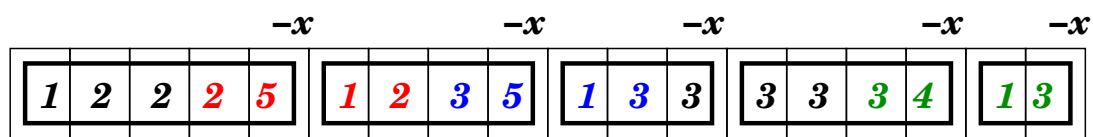
Theorem 5.

$$\mathcal{F}^{\mathbb{P}}(\mathbf{x}, \mathbf{z}, t)) = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S|=n} XZ(S)}.$$

(4)



$$u = 2312 \quad k = 5$$



$C(B, w, L)$

$1 \ 2 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 3 \ 3 \ 3 \ 3$

$$\begin{array}{c}
 \mathbf{2 \ 2} \longrightarrow \mathbf{2 \ 3 \ 1 \ 2} \quad \mathbf{2 \ 4 \ 1 \ 2} \quad \mathbf{2 \ 5 \ 1 \ 2} \\
 1 - x \quad -x z_3 z_1 t^2 \quad -x z_4 z_1 t^2 \quad -x z_5 z_1 t^2
 \end{array}$$

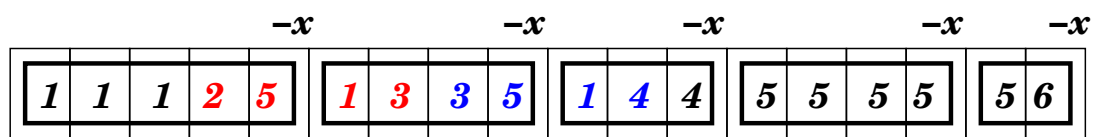
$$\begin{array}{c}
 \mathbf{3 \ 3} \longrightarrow \mathbf{3 \ 4 \ 1 \ 3} \quad \mathbf{3 \ 5 \ 1 \ 3} \quad \mathbf{3 \ 4 \ 2 \ 3} \quad \mathbf{3 \ 5 \ 2 \ 3} \\
 1 - x \quad -x z_4 z_1 t^2 \quad -x z_4 z_1 t^2 \quad -x z_4 z_2 t^2 \quad -x z_5 z_2 t^2
 \end{array}$$

$$\mathbf{1 \ 1} \longrightarrow 1 - x$$

Theorem 6.

$$\sum_{w \in \mathbb{P}^*, \text{des}(w)=0} t^{|w|} z^w \prod_i x_i^{\text{ii}(w)} = 1 + \sum_{S \subseteq \mathbb{P}, S \neq \emptyset} \prod_{i \in S} \frac{tz_i}{1 - tz_i x_i}.$$

$$u = 2413 \quad k = 6$$



$C(B, w, L)$

$$1 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 4 \ 4 \ 5 \ 5 \ 5 \ 5 \ 5 \ 6$$

$$\begin{array}{c}
 \mathbf{2 \ 3} \longrightarrow \mathbf{2 \ 4 \ 1 \ 3} \qquad \mathbf{2 \ 5 \ 1 \ 3} \qquad \mathbf{2 \ 6 \ 1 \ 3} \\
 1 - x \quad -x z_4 z_1 t^2 \quad -x z_5 z_1 t^2 \quad -x z_6 z_1 t^2
 \end{array}$$

$$\begin{array}{c}
 \mathbf{3 \ 4} \longrightarrow \mathbf{3 \ 5 \ 1 \ 4} \qquad \mathbf{3 \ 5 \ 2 \ 4} \qquad \mathbf{3 \ 6 \ 1 \ 4} \qquad \mathbf{3 \ 6 \ 2 \ 4} \\
 1 - x \quad -x z_5 z_1 t^2 \quad -x z_6 z_2 t^2 \quad -x z_6 z_1 t^2 \quad -x z_6 z_2 t^2
 \end{array}$$

$$\mathbf{1 \ 2} \longrightarrow 1 - x$$

Theorem 7.

$$\sum_{w \in \mathbb{P}^*, \text{des}(w)=0} t^{|w|} z^w \prod_{i < j} y_{ij}^{\text{ij}(w)} = 1 + \sum_{j \geq 1} \frac{z_j t}{1 - z_j t} + \sum_{S \subseteq \mathbb{P}, |S| \geq 2} YZT(S).$$

Given a set $S = \{i_1 < i_2 < \cdots < i_j\} \subseteq \mathbb{P}$, we let

$$YZT(S) = y_{i_1 i_2} y_{i_2 i_3} \cdots y_{i_{j-1} i_j} \prod_{r=1}^j \frac{z_{i_r} t}{1 - z_{i_r} t}. \quad (5)$$