

Pattern Avoidance on k -ary Heaps

(Work in Progress - aren't they all)

Derek Levin, Lara Pudwell, Manda Riehl, and Andrew Sandberg

University of Wisconsin - Eau Claire, Valparaiso University

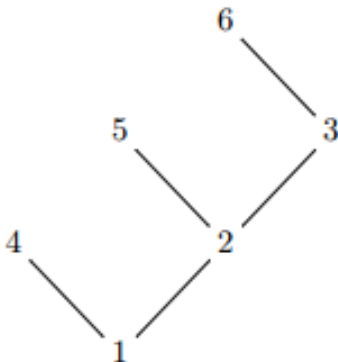
Permutation Patterns - July 10, 2014

Once upon a time. . .



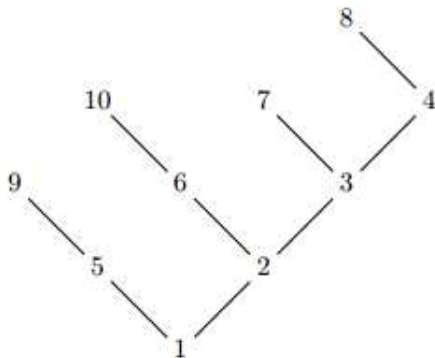
Sophia Yakoubov: Paris 2013

Sophia Yakoubov: Paris 2013 Pattern Avoidance on Combs



[1]

Motivation

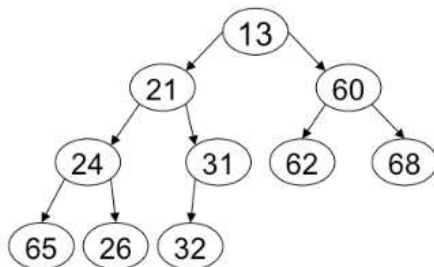


Something like combs, but not combs

Heaps!

Something like combs, but not combs

Heaps!



Definition

A *complete binary tree* is a tree where each node has 2 or fewer children, all levels except possibly the last are completely full, and the last level has all its nodes to the left side.

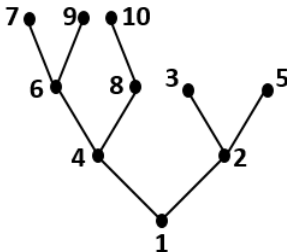
Definition

A *heap* is a complete binary tree labelled with $\{1, \dots, n\}$ such that every child has a larger label than its parent.

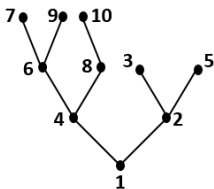
Heaps

Definition

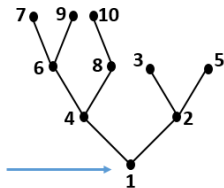
A *heap* is a complete binary tree labelled with $\{1, \dots, n\}$ such that every child has a larger label than its parent.



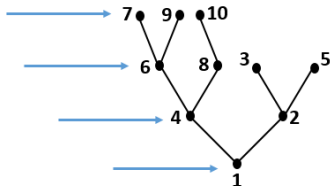
Where's the pattern?



Where's the pattern?

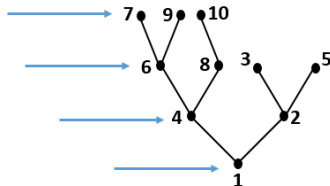


Where's the pattern?



1 4 2 6 8 3 5 7 9 10

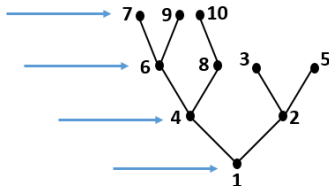
Where's the pattern?



1 4 2 6 8 3 5 7 9 10

Notice this heap contains 123, 132, 213, 231, 312.

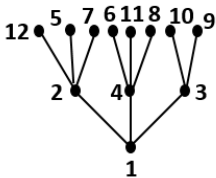
Where's the pattern?



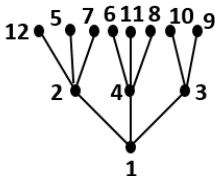
1 4 2 6 8 3 5 7 9 10

Notice this heap contains 123, 132, 213, 231, 312.
But it avoids 321.

k -ary Heaps

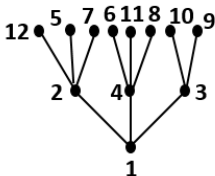


k -ary Heaps



1 2 4 3 12 5 7 6 11 8 10 9

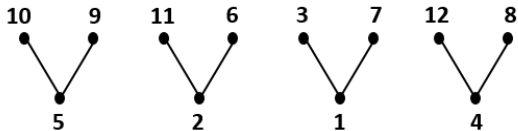
k -ary Heaps



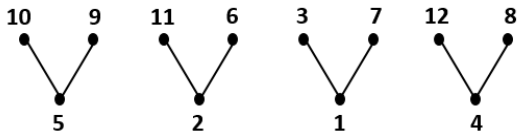
1 2 4 3 12 5 7 6 11 8 10 9

This heap avoids 231.

Forests of Heaps

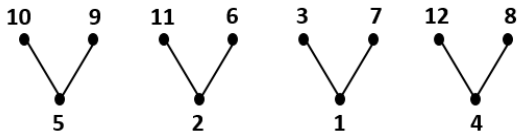


Forests of Heaps



5 10 9 2 11 6 1 3 7 4 12 8

Forests of Heaps



5 10 9 2 11 6 1 3 7 4 12 8

Notice each tree avoids 213, 231, 312, 321, but the forest contains 213, 231, 312, 321.

Crunch the numbers, cross your fingers

Heaps Avoid- ing:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	
123	1, 1, 1, 0, 0, 0, 0, 0...	
132	1, 1, 1, 1, 1, 1, 1, 1...	
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	
$\{213, 231\} =$ $\{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	
$\{231, 312\} =$ $\{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	

Crunch the numbers, cross your fingers

Heaps Avoid- ing:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0...	
132	1, 1, 1, 1, 1, 1, 1, 1...	
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	
$\{213, 231\} =$ $\{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	
$\{231, 312\} =$ $\{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	

Crunch the numbers, cross your fingers

Heaps Avoid- ing:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0...	A000004
132	1, 1, 1, 1, 1, 1, 1, 1...	A000012
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	
$\{213, 231\} =$ $\{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	
$\{231, 312\} =$ $\{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	

Crunch the numbers, cross your fingers

Heaps Avoid- ing:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0...	A000004
132	1, 1, 1, 1, 1, 1, 1, 1...	A000012
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	A208355
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	
$\{213, 231\} =$ $\{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	
$\{231, 312\} =$ $\{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	

Crunch the numbers, cross your fingers

Heaps Avoid- ing:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0...	A000004
132	1, 1, 1, 1, 1, 1, 1, 1...	A000012
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	A208355
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	
$\{213, 231\} = \{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	A016116
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	
$\{231, 312\} = \{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	

Crunch the numbers, cross your fingers

Heaps Avoid- ing:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0...	A000004
132	1, 1, 1, 1, 1, 1, 1, 1...	A000012
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	A208355
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	
$\{213, 231\} =$ $\{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	A016116
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	
$\{231, 312\} =$ $\{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	A000045

Heaps Avoiding (231, 312)

Narayana-Zidek-Capell Numbers:

1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, ...

Given by relation:

$$a_n = 2a_{n-1} \text{ if } n \text{ even}$$

$$a_n = 2a_{n-1} - a_{\frac{n-1}{2}} \text{ if } n \text{ odd}$$

Heaps Avoiding (231, 312)

Narayana-Zidek-Capell Numbers:

1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, ...

Insertion argument:

Heaps Avoiding (231, 312)

Narayana-Zidek-Capell Numbers:

1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, ...

Insertion argument:

Insert $n + 1$, but leave it an increasing tree

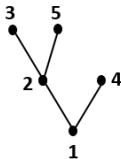
Heaps Avoiding (231, 312)

Narayana-Zidek-Capell Numbers:

1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, ...

Insertion argument:

Insert $n + 1$, but leave it an increasing tree



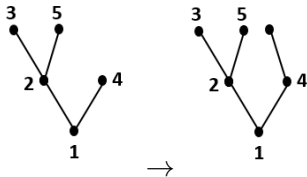
Heaps Avoiding (231, 312)

Narayana-Zidek-Capell Numbers:

1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, ...

Insertion argument:

Insert $n + 1$, but leave it an increasing tree



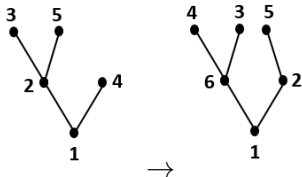
Heaps Avoiding (231, 312)

Narayana-Zidek-Capell Numbers:

1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, ...

Insertion argument:

Insert $n + 1$, but leave it an increasing tree



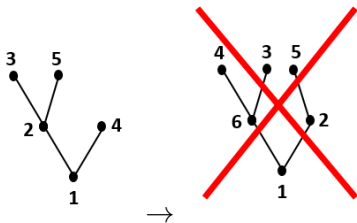
Heaps Avoiding (231, 312)

Narayana-Zidek-Capell Numbers:

1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, ...

Insertion argument:

Insert $n + 1$, but leave it an increasing tree



Heaps Avoiding (231, 312)

Lemma

The vertex labelled n is always a leaf. After insertion, the vertex labelled $n + 1$ is always a leaf.

Heaps Avoiding (231, 312)

Lemma

In order to avoid 231 and 312, $n + 1$ must be inserted directly before n or at the end.

Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

Proof of Lemma:

- $n + 1$ at last leaf: OK
- $n + 1$ right before n : OK

Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

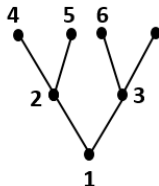
Proof of Lemma:

- If $n + 1$ is inserted more before n , we create a 312.

Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

Proof of Lemma:

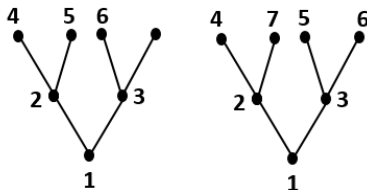
- If $n + 1$ is inserted more before n , we create a 312.



Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

Proof of Lemma:

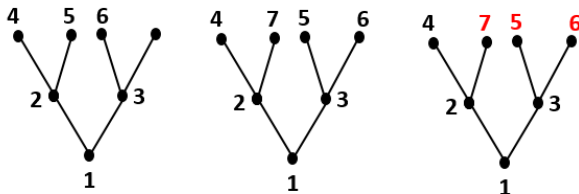
- If $n + 1$ is inserted more before n , we create a 312.



Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

Proof of Lemma:

- If $n + 1$ is inserted more before n , we create a 312.



Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

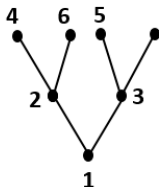
Proof of Lemma:

- If $n + 1$ is inserted after n , but not at the end, we create a 231.

Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

Proof of Lemma:

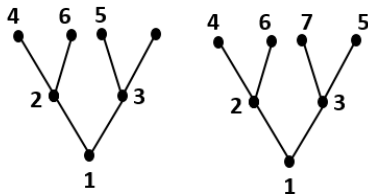
- If $n+1$ is inserted after n , but not at the end, we create a 231.



Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

Proof of Lemma:

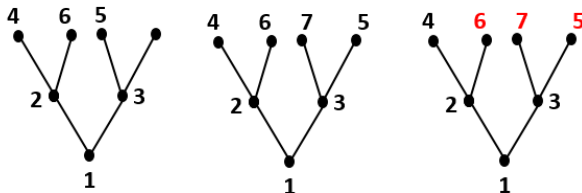
- If $n + 1$ is inserted after n , but not at the end, we create a 231.



Heaps Avoiding (231, 312): $n + 1$ must be right before n , or at end

Proof of Lemma:

- If $n + 1$ is inserted after n , but not at the end, we create a 231.



Heaps Avoiding (231, 312)

Easy Case: n is even. So the new leaf is the sibling of a current leaf. Internal nodes stay internal, leaves stay leaves.

Heaps Avoiding (231, 312)

Easy Case: n is even. So the new leaf is the sibling of a current leaf. Internal nodes stay internal, leaves stay leaves.

We can put $n + 1$ at the (new) last leaf, or we can insert it right before n and push everything along.

Heaps Avoiding (231, 312)

Easy Case: n is even. So the new leaf is the sibling of a current leaf. Internal nodes stay internal, leaves stay leaves.

We can put $n + 1$ at the (new) last leaf, or we can insert it right before n and push everything along.

$$a_n = 2a_{n-1}.$$

Heaps Avoiding (231, 312)

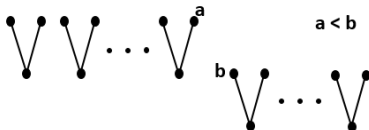
Second Case: n is odd. So the new leaf is child of a former leaf.
Is it possible that we push a small label to be a child over a larger label that was a leaf?

Heaps Avoiding (231, 312)

Inserting $n + 1$ at the last leaf:
Still OK!

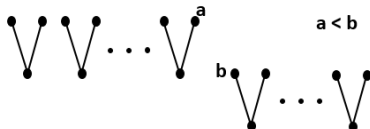
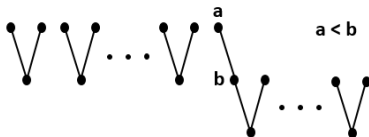
Heaps Avoiding (231, 312)

Inserting $n + 1$ anywhere except the first or last leaf:



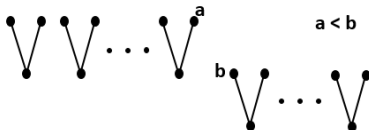
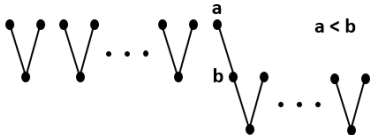
Heaps Avoiding (231, 312)

Inserting $n + 1$ anywhere except the first or last leaf:



Heaps Avoiding (231, 312)

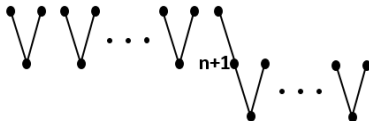
Inserting $n + 1$ anywhere except the first or last leaf:



We must already have had a 231, namely $b n a$.

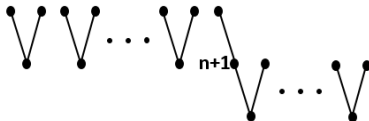
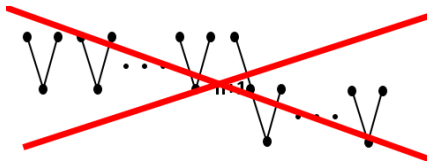
Heaps Avoiding (231, 312)

Inserting $n + 1$ at the first leaf:



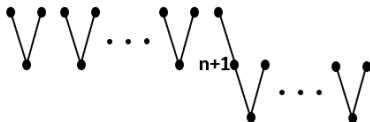
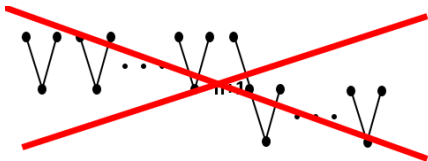
Heaps Avoiding (231, 312)

Inserting $n + 1$ at the first leaf:



Heaps Avoiding (231, 312)

Inserting $n + 1$ at the first leaf:



We don't want to count this, even though it may still avoid 231, 312.

Heaps Avoiding (231, 312)

How many shouldn't we count?

Since n was on the first leaf, all other leaf labels are in decreasing order.

Heaps Avoiding (231, 312)

How many shouldn't we count?

Since n was on the first leaf, all other leaf labels are in decreasing order.

The subtree obtained by removing all leaves needs to avoid 231,312.

Heaps Avoiding (231, 312)

How many shouldn't we count?

Since n was on the first leaf, all other leaf labels are in decreasing order.

The subtree obtained by removing all leaves needs to avoid 231, 312.

There are $\frac{n-1}{2}$ nodes on that subtree.

Heaps Avoiding (231, 312)

How many shouldn't we count?

Since n was on the first leaf, all other leaf labels are in decreasing order.

The subtree obtained by removing all leaves needs to avoid 231,312.

There are $\frac{n-1}{2}$ nodes on that subtree.

Thus $a_n = 2a_{n-1} - a_{\frac{n-1}{2}}$ when n is odd.

Heaps Avoiding (231, 312)

How many shouldn't we count?

Since n was on the first leaf, all other leaf labels are in decreasing order.

The subtree obtained by removing all leaves needs to avoid 231, 312.

There are $\frac{n-1}{2}$ nodes on that subtree.

Thus $a_n = 2a_{n-1} - a_{\frac{n-1}{2}}$ when n is odd.

So we have the same recurrence relation as the Narayana-Zidek-Capell numbers.

Heaps Avoiding 231

Test yourself!

Heaps Avoiding 231

- 1

Heaps Avoiding 231

•

1

2

•

•

1

Heaps Avoiding 231

•

1

2

•

•

1

2

•

3

•

•

1

3

•

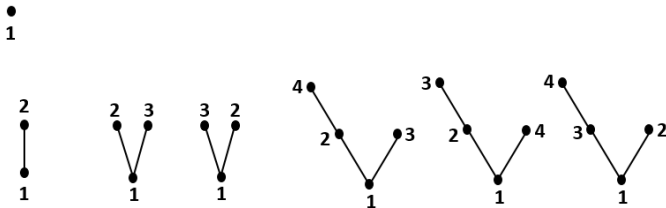
2

•

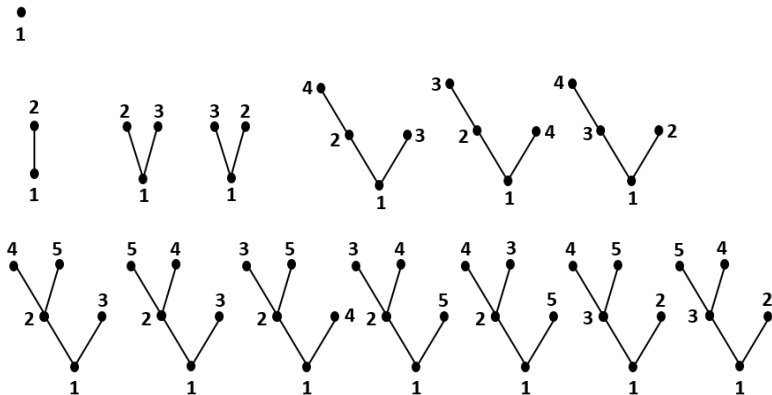
•

1

Heaps Avoiding 231



Heaps Avoiding 231



This site is supported by donations to [The OEIS Foundation](#).

the on-line encyclopedia of integer sequences®

founded in 1964 by N. J. A. Sloane

Many excellent [designs](#) for a new banner were submitted. We will use the best of them in rotation.

[Hints](#)

(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

Search: seq:1,1,2,3,7,14,37,80,222

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the [form provided](#) and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Search completed in 0.100 seconds

[Lookup](#) | [Welcome](#) | [Wiki](#) | [Register](#) | [Music](#) | [Plot 2](#) | [Demos](#) | [Index](#) | [Browse](#) | [More](#) | [WebCam](#)
[Contribute new seq. or comment](#) | [Format](#) | [Transforms](#) | [Superseeker](#) | [Recent](#) | [More pages](#)
[The OEIS Community](#) | Maintained by [The OEIS Foundation Inc.](#)

Content is available under [The OEIS End-User License Agreement](#).

Last modified July 8 12:42 EDT 2014. Contains 244265 sequences.

Let's look at where the n appears. (Definitely on a leaf)

Let's look at where the n appears. (Definitely on a leaf)

All the labels before n are less than all labels after n .

Let's look at where the n appears. (Definitely on a leaf)

All the labels before n are less than all labels after n .

The labels on the leaves after n can be arranged in Catalan many ways.

Let's look at where the n appears. (Definitely on a leaf)

All the labels before n are less than all labels after n .

The labels on the leaves after n can be arranged in Catalan many ways.

The subheap before n is a smaller case of a heap avoiding 231.

Let b_n be the number of heaps with n nodes avoiding 231.

Let b_n be the number of heaps with n nodes avoiding 231.

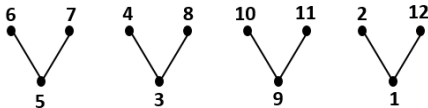
Let i be the number of leaves after n . $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

Let b_n be the number of heaps with n nodes avoiding 231.

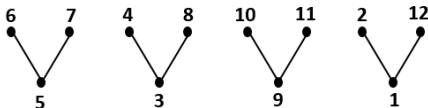
Let i be the number of leaves after n . $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

$$b_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} C_i b_{n-i-1}$$

k -ary forests avoiding 132

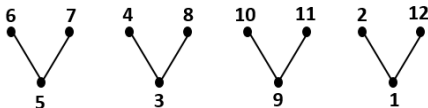


k -ary forests avoiding 132



Obvious statements: Each tree avoids 132.

k -ary forests avoiding 132



Obvious statements: Each tree avoids 132.

The roots of each tree avoid 132.

k -ary forests avoiding 132



k -ary forests avoiding 132



Lemma

Knowing the roots is enough!

Theorem

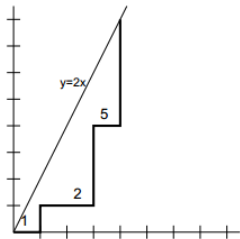
The number of forests of t heaps each with v vertices that avoid 132 is given by $\frac{1}{vt+1} \binom{(v+1)t}{t}$.

Theorem

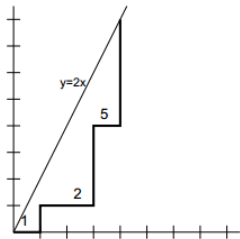
The number of forests of t heaps each with v vertices that avoid 132 is given by $\frac{1}{vt+1} \binom{(v+1)t}{t}$.

Proof method: Bijection to the number of paths under the line $y = vx$ from $(0, 0)$ to (t, vt) using steps $(0, 1)$ and $(1, 0)$.

Examples of bijection

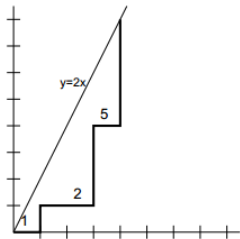


Examples of bijection



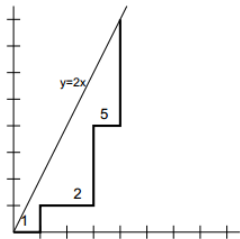
Make a string of the level step heights in reverse order: 4 1 1 0

Examples of bijection



Make a string of the level step heights in reverse order: 4 1 1 0
Add one to each element of the string. 5 2 2 1

Examples of bijection

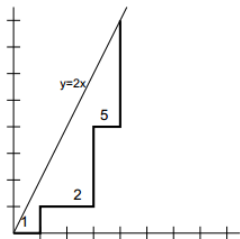


Make a string of the level step heights in reverse order: 4 1 1 0

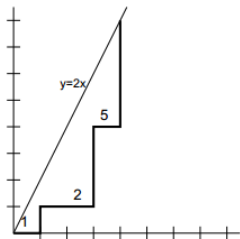
Add one to each element of the string. 5 2 2 1

The smallest number currently unused in the forest that is greater than or equal to the next element of the string gives the next root.

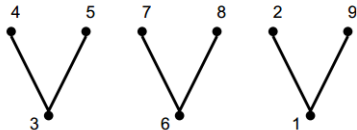
Examples of bijection



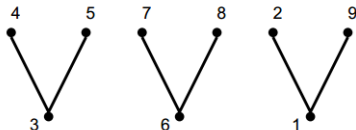
Examples of bijection



Examples of bijection



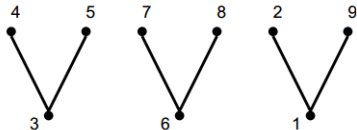
Examples of bijection



Make a string, starting with the root of the first heap.

3

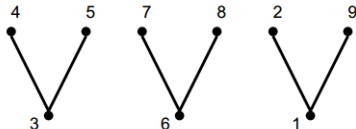
Examples of bijection



Make a string, starting with the root of the first heap.
Repeat the same number in the string for each root as long as the
permutation is increasing.

$3 \rightarrow 33$

Examples of bijection



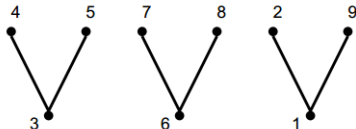
Make a string, starting with the root of the first heap.

Repeat the same number in the string for each root as long as the permutation is increasing.

If the permutation has a descent, the next root is the next entry in the string.

$3 \rightarrow 3 \ 3 \rightarrow 3 \ 3 \ 1$

Examples of bijection



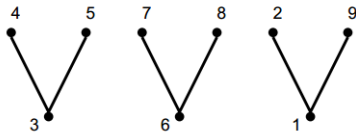
Make a string, starting with the root of the first heap.

Repeat the same number in the string for each root as long as the permutation is increasing.

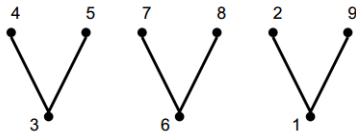
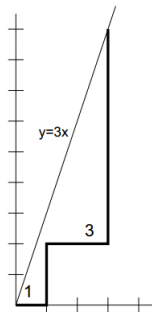
If the permutation has a descent, the next root is the next entry in the string.

Subtract 1 from each element of the string. These are your sequence of level steps. $3 \rightarrow 3$ $3 \rightarrow 3$ 3 $1 \rightarrow 2$ 2 0

Examples of bijection



Examples of bijection



Corollary

Let σ be a permutation of length nm composed of a concatenation of m increasing sequences of length n . The number of such σ that avoid 132 is $\frac{1}{nm+1} \binom{(n+1)m}{nm+1}$.

Summary

Heaps Avoiding:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0...	A000004
132	1, 1, 1, 1, 1, 1, 1, 1...	A000012
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	A208355
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	Soon in OEIS!
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	OPEN
$\{213, 231\} = \{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	A016116
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	A000124
$\{231, 312\} = \{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	A002083
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	A000045

What about k -ary?

Heaps Avoid- ing:	Sequence	OEIS#
$\emptyset$	1, 1, 2, 3, 8, 20, 80, 210, 896...	A056971
123	1, 1, 1, 0, 0, 0, 0, 0...	A000004
132	1, 1, 1, 1, 1, 1, 1, 1...	A000012
213	1, 1, 2, 2, 5, 5, 14, 14, 42...	A208355
231 = 312	1, 1, 2, 3, 7, 14, 37, 80, 222, ...	Soon to be in OEIS!
321	1, 1, 2, 3, 7, 16, 45, 111, 318...	OPEN
$\{213, 231\} =$ $\{213, 312\}$	1, 1, 2, 2, 4, 4, 8, 8, 16...	A016116
$\{213, 321\}$	1, 1, 2, 2, 4, 4, 7, 7, 11...	A000124
$\{231, 312\} =$ $\{231, 321\}$	1, 1, 2, 3, 6, 11, 22, 42, 84...	A002083
$\{231, 312, 321\}$	1, 1, 2, 3, 5, 8, 13...	A000045

What next?

- Trees that aren't heaps

What next?

- Trees that aren't heaps
- Unary-binary, binary, k -ary

What next?

- Trees that aren't heaps
- Unary-binary, binary, k -ary
- Slightly different question: How many permutations avoid σ can be realized as trees?

Thank You!

- Anant Godbole
- Permutation Patterns 2014 Organizers
- UWEC Department of Mathematics
- UWEC Office of Research and Sponsored Programs

- [1] Yakoubov, Sophia. Pattern Avoidance in Extensions of Comb-Like Posets. arXiv:1310.2979v2 [math.CO]
- [2] Lewis, Joel. Pattern Avoidance for Alternating Permutations and Reading Words of Tableaux. Department of Mathematics MIT (2012): 1-69. June 2012. Web. 22 Aug. 2013
- [3] R. Simion and F.W. Schmidt, Restricted Permutations, Europ. J. Comb., 6, 1985, 383-406.